BY PROF. G. A. GIBSON

INTRODUCTION TO THE CALCULUS

ELEMENTARY TREATISE ON THE CALCULUS

ADVANCED CALCULUS

BY PROF. G. A. GIBSON AND DR. P. PINKERTON

ELEMENTS OF ANALYTICAL GEOMETRY

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AN INTRODUCTION TO THE CALCULUS

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BASED ON GRAPHICAL METHODS

BY

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PREFACE.

In the following pages an attempt has been made to present the elements of the Calculus in a way that will appeal to comparatively immature students. The reasoning is based essentially on the graphical representation of a function, and it is hoped that students who have obtained a firm grasp of the meaning of such representation will be able to gain, without serious difficulty, a right apprehension of the meaning of a differential coefficient. I have deliberately abstained from crowding the book with diagrams, because I assume that before the student begins the study of the Calculus he will be quite familiar with the methods of graphing the ordinary functions and will have reached the stage at which he can at once form a mental picture of such graphs. From the educational point of view the graph is not so much an end in itself as an aid to the comprehension of the variation of a function.

In all practical applications of the Calculus the consideration of a differential coefficient as the measure of a rate of variation is of the utmost importance, and this aspect of a differential coefficient can be very readily understood by any one who is familiar with graphical work. The only satisfactory way, however, by which the conception of a rate can be adapted to practical uses is, in my judgment, the method of limits; I have therefore used that method throughout the book.

The first eight chapters treat of algebraic functions alone, and they contain many of the simpler applications of the Calculus; the chapter on Graphical Integration may, if desired, be taken up as soon as these chapters have been mastered. The discussion of the circular and exponential functions will probably be found distinctly harder; but long experience has convinced me that the difficulties are due not to the nature of the Calculus but to the student's imperfect knowledge of trigonometry. It seems to me that a student who has not a firm grasp of the Addition Theorem should postpone his study of the differentiation and integration of circular functions till he has obtained that grasp; the Theorem is absolutely necessary for many of the most important applications of trigonometry and is, after all, very easy to understand and apply.

I have included a short discussion of the Fourier Series, because of the numerous applications it is now receiving in elementary work; in the sections that treat of the decomposition of an empirical function I have given a solution, due to Professor Runge, which seems to me to be thoroughly practical.

It may seem to many that I have tried to build on too slight a foundation of elementary mathematics. I am well aware that a thorough knowledge of the Calculus can only be obtained by those who have had a broader training in the elements of mathematics than I assume in my readers; but I have for several years given courses on the Calculus to large classes of evening students and I have found that it is quite possible to do much good work on the lines followed in this book. A student, however, who wishes to profit by the course laid down must work many of the examples in the various sets of Exercises; the principal results in differentiation and integration must be as familiar

as the multiplication table, and the best way of acquiring that familiarity is by working numerous examples. The time required for this purpose is by no means so great as many people suppose; in any case, that familiarity must be acquired if the Calculus is to be the instrument and not the master of the student.

In conclusion I desire to tender my cordial thanks to several friends who have encouraged me in the production of the book: to Mr. P. Bennett, Mr. W. A. Lindsay, Mr. P. Pinkerton and Mr. A. T. Simmons for very efficient help in proof-reading; to Mr. J. Dougall and Mr. J. Miller for the solutions of the examples; and to Sir Richard Gregory for his most helpful advice at all stages of the progress of the book. I must also thank the printing staff of Messrs. MacLehose for the excellence of their share of the work.

GEORGE A. GIBSON.

To this reprint I have added a chapter on Maclaurin's Theorem and Differential Equations. In the treatment of Maclaurin's Theorem my object has been to make the student familiar with its more useful applications rather than to exercise him in rigorous mathematical proof; for the beginner it is more important to be able to expand a function and to test whether the series is convergent than to prove the conditions under which an expansion is possible. The types of Differential Equations discussed are such as occur most frequently in everyday work; special attention is given to linear equations of the second order with constant coefficients. It is hoped that the Note on

Partial Differentiation will be of some use to elementary students who have wisdom enough to extend their reading beyond the bare minimum of Engineering Physics.

For the collection of Miscellaneous Exercises I am indebted to Dr. John M'Whan, Lecturer in Mathematics in this University; I am also indebted to him for a careful revision of the MS. and the proofs of the new matter.

THE UNIVERSITY,
GLASGOW, September, 1916.

CONTENTS.

CHAPTER I.

INTRODUCTION.

CHAPTER II.

DIFFERENTIATION OF POWERS. MAXIMA AND MINIMA.

CHAPTER III.

DIFFERENTIALS. HIGHER DERIVATIVES.

CHAPTER IV.

APPLICATIONS TO MECHANICS. FURTHER THEOREMS ON DIFFERENTIATION.

CHAPTER V.

INTEGRATION OF POWERS.

CHAPTER VI.

AREAS. DEFINITE INTEGRALS.

CHAPTER VII.

INTEGRAL AS LIMIT OF A SUM. SIMPSON'S RULES.

CHAPTER VIII.

APPLICATIONS TO MECHANICS.

CHAPTER IX.

DIFFERENTIATION OF DIRECT TRIGONOMETRIC FUNCTIONS.

CHAPTER X.

INTEGRATION OF DIRECT TRIGONOMETRIC FUNCTIONS. MEAN VALUES.

CHAPTER XI.

FOURIER SERIES.

CHAPTER XII.

INVERSE CIRCULAR FUNCTIONS. LOGARITHMIC AND EXPONENTIAL FUNCTIONS.

CHAPTER XIII.

APPLICATIONS. CURVATURE. BENDING OF BEAMS. CATENARY. ALTERNATE CURRENTS. DOUBLE INTEGRALS.

CHAPTER XIV.

GRAPHICAL INTEGRATION.

CHAPTER XV.

MACLAURIN'S THEOREM. DIFFERENTIAL EQUATIONS.

NOTE.—In the first eight Chapters Algebraic Functions alone are considered; the Chapter on Graphical Integration may be read, if desired, as soon as these Chapters have been mastered. Articles 63-67 may also be taken before Chapter X.; Article 63 may be taken immediately after Chapter IV.

CHAPTER I.

INTRODUCTION.

1. Graphs. In the following pages the processes of differentiation and integration are so closely connected with the graphical representation of functions that it seems proper to begin with a short reference to graphical methods. These methods, however, have in recent years been so widely adopted, even in elementary teaching, that a very brief reference to them will be sufficient. In this chapter we summarise the more important properties of a graph and explain the meaning of various technical terms that are constantly used.

If pairs of numbers are chosen at random and the points plotted which have these numbers as coordinates, there will be no orderly arrangement among the points: they will be scattered all over the diagram. If, however, the coordinates are connected by an equation the case is altered; the points will now be arranged in a manner that suggests a definite curve on which they all lie. This curve is called **the graph of the equation** while, in reference to the curve, the equation is called **the equation of the graph.** Other phrases are often used to express the connection between the equation and the graph. Thus, the equation is said to be **represented** by the graph, while the graph is said to be **given** by the equation; the curve is, so to speak, the geometrical counterpart, or picture, of the equation.

In graphical work a point is specified by its coordinates, and the point is said to be *given* when its coordinates are

y is now said to be expressed *explicitly* as a function of x, while, so long as the equation is not solved for y, it is only *implicitly* expressed as a function of x; in the unsolved form of the equation y is said to be an *implicit function* of x, while in the solved form it is said to be an *explicit function* of x.

The equation also defines x as a function of y, namely,

$$x = \frac{7y-3}{5y-4},$$

as may be seen by solving the equation for x. Both functions are *fractional* functions of their arguments.

Example 2. A stone is thrown vertically upwards with a velocity of V feet per second; assuming that the resistance of the air may be left out of account, express the distance travelled in a given time as a function of the time.

Suppose that in t seconds the stone has risen s feet above the point of projection; then it is shown in books on mechanics that

$$s = Vt - \tfrac{1}{2}gt^2,$$

where g is a constant, equal to 32·2 approximately. The distance travelled is therefore a function of the time; since the time t enters into the expression of the function in the second and no higher degree, the distance s is a *quadratic function* of the time t.

The velocity v at time t is a linear function of the time, because

$$v = V - gt.$$

In this example s, t, v are variables; V, g are constants.

Example 3. A point moves in a circle of radius 5 with its centre at the origin of coordinates; express the ordinate of the point as a function of its abscissa.

Let x, y be the coordinates of the point in any one of its positions;

then $$x^2 + y^2 = 25, \quad\text{.........(i)}$$

and therefore $$y = \sqrt{(25 - x^2)}. \quad\text{.........(ii)}$$

To express y fully, we must remember that the root may be either positive or negative; the symbol $\sqrt{(25-x^2)}$ is *two-valued*, namely, is either $+\sqrt{(25-x^2)}$ or $-\sqrt{(25-x^2)}$. The $+$ sign goes with points above the x-axis, the $-$ sign with points below that axis.

Equation (i) defines y implicitly as a function of x. It often happens, as in this case, that there correspond two (or more) values of the function to any one value of the argument. We must, of course, restrict ourselves to one of these values at a time; we may in fact consider y to be made up of the two functions

$$y = +\sqrt{(25-x^2)} \text{ and } y = -\sqrt{(25-x^2)},$$

each of which is *single-valued*, that is, has only one value of y for any one value of x.

It is also to be noted that y is only defined for values of x from $x = -5$ to $x = 5$. For values of x greater (numerically) than 5 the values of y are imaginary.

4. Inverse Functions. As we have seen in §3, example 1, an equation between x and y not only defines y as a function of x but also defines x as a function of y. Two functions thus defined by one equation are said to be **inverse** to each other. Thus, as another example, the equation $y=x^3$, when solved for x, gives $x=\sqrt[3]{y}$, and therefore defines two functions which are inverse to each other, namely *the cube* and *the cube root.*

A function and its inverse are represented by the same graph, but when the inverse function is not single-valued care must be taken to select the proper part of the graph when considering the inverse function. We shall return to this matter when we come to discuss the differentiation of the inverse circular functions.

5. Gradients. The coefficient a of x in the equation

$$y=ax+b \quad \text{.........................(1)}$$

is called **the gradient** (sometimes, **the slope**) of the straight line given by the equation.

The following ways of interpreting the gradient are important.

Geometrically, the x-axis being supposed horizontal and the y-axis vertical, the gradient a measures **the rate at which the line rises or falls.**

When a is positive, the line has a right-hand upward slope; a point rises as it moves to the right along the line.

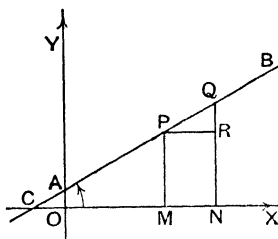


Fig. 1.

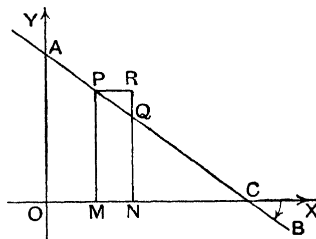


Fig. 2.

In Fig. 1 the gradient is OA/CO or RQ/PR; the rise RQ is a times the horizontal advance PR.

When a is negative, the line has a right-hand downward

slope; a point falls as it moves to the right along the line. In Fig. 2 the gradient is AO/OC or RQ/PR; the fall RQ is $-a$ times the horizontal advance PR.

When $a=0$ the line is horizontal. The greater a is (numerically) the greater is the angle the line makes with the horizontal; when the angle is 90° the gradient is said to be infinite.

Trigonometrically, the gradient a is **the tangent of the angle which the line makes with the x-axis.**

When the line has a right-hand downward slope (Fig. 2), the angle may be taken to be the *negative* angle XCB or the *obtuse* angle XCA; $\tan XCB$ and $\tan XCA$ are both negative. In graphical work it is usually better to take the angle of the gradient as negative in this case.

Algebraically, the gradient a measures **the rate at which the function $ax+b$ varies as x varies.**

If x increases from any value x_1 to the value x_1+h, then y changes from ax_1+b to $a(x_1+h)+b$; the *increase* of y is ah, that is, a times the *increase* of x. The increase of $ax+b$ is thus always in simple proportion to the increase of x, and the linear function is therefore called a **uniformly varying** function of its argument; the **rate** at which the function changes is **constant** and equal to a. If a is negative, y decreases as x increases; a decrease is considered to be a negative increase.

The gradient of any straight line which is at right angles to the line given by equation (1) is $-1/a$.

6. Increments. As a point moves from P to Q along the line AB (Figs. 1, 2) the amount MN or PR by which the x of the point changes is usually called the **increment** of x. When x takes the increment MN or PR, y takes the increment RQ; in Fig. 2 the step RQ is negative and the increment is expressed by a negative number.

In all cases the gradient of a straight line is obtained by dividing any increment of the ordinate by the corresponding increment of the abscissa. When the quotient is positive, the ordinate increases as the abscissa increases; the function represented by the ordinate is then an **increasing function** of its argument. When the quotient is negative, the ordinate

decreases as the abscissa increases; the function represented by the ordinate is then a **decreasing function** of its argument.

7. Average Gradient. Rates. The gradient of a straight line is the same for every portion of it, long or short. If, however, we take two points P and Q on a curved line (Fig. 3) and divide the increment RQ which y takes, as a point moves along the curve from P to Q, by the corresponding increment PR of x, the quotient obtained will manifestly depend on the positions both of P and of Q. We must therefore consider what is meant when we speak of the gradient of a curved line.

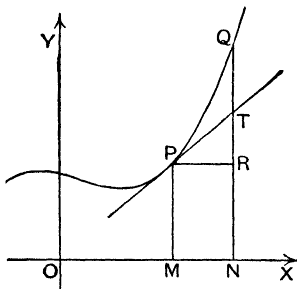


Fig. 3.

The increment of the ordinate of a point which moves from P to Q is the same whether the point travels along the arc PQ or along the chord PQ; hence the gradient of the chord PQ, namely RQ/PR, is called **the average gradient of the arc** PQ. When Q is very close to P, the direction of the chord PQ will differ very little from that of the tangent PT to the curve at P; the closer Q is to P the less does the direction of the chord PQ differ from that of the tangent PT. The gradient of the tangent PT is therefore taken as the gradient of the curve *at* P.

If MP, NQ denote two values of the function represented by the curve (Fig. 3), then the gradient of the chord PQ measures **the average rate** at which the function changes as its argument changes from OM to ON. (It is easy to see

that if, as the argument changes from OM to ON, the function were to change at the uniform rate RQ/PR it would receive the increment RQ, which is the increment it actually takes.) Just as the gradient of the tangent PT is taken as the gradient of the curve *at* P, so the gradient of the tangent PT is taken as the rate at which the function is changing when its argument is equal to OM, the abscissa of P.

A rate of change is *the ratio* of two amounts of change. When one magnitude is a function of another, any increase in this other magnitude will produce an increase, or a decrease, in the function; the ratio of the change in the function to the corresponding change in the argument is the number which has been defined as the average rate of change for that change of the argument.

For the linear function $ax+b$, the average rate of change is the same whatever be the value, x_1 say, *from which* the argument changes, and whatever be the amount, h say, *by which* the argument changes; the ratio of the two amounts of change is always a, and therefore the linear function is one which changes at a constant, or uniform, rate.

In general, however, the average rate of change of a function depends both on the value of the argument from which the change begins and on the amount by which the argument changes. When the amount by which the argument changes is very small, the average rate will clearly give a good approximation to the rate at which the function is changing for that value of the argument from which the change begins; the smaller the change in the argument the better will be the measure of the rate.

A rate of change, then, always implies two variables; an independent (change causing) variable, or argument, and a dependent variable, or function. The rate of change of the function is a rate "with respect to" or "relative to" its argument; for brevity, however, the phrase specifying the independent variable is often omitted. When the argument is specified, such phrases as "the x-rate of change," "the x-gradient," "the t-rate of change," "the t-gradient" are often employed, according as the argument is x or t.

In calculating gradients we shall usually suppose the abscissa (or argument) to increase algebraically; in other words, the increment of the abscissa will be taken to be positive. The sign of the corresponding increment of the ordinate (or function) will show whether the ordinate increases or decreases for this increment of the abscissa. There is, of course, no reason except that of convenience for choosing the increment of the abscissa to be positive.

Example 1. Find the average gradient of the graph of $y=x^2$, (i) as x increases from 2 to 2·5, (ii) as x increases from 2 to $2+h$.

(i) When $x=2$, $y=4$, and when $x=2{\cdot}5$, $y=6{\cdot}25$. The increment of x is 0·5, and the corresponding increment of y is 2·25. Therefore,

$$\text{av. grad.}=\frac{(2{\cdot}5)^2-2^2}{2{\cdot}5-2}=\frac{2{\cdot}25}{0{\cdot}5}=4{\cdot}5.$$

(ii) When $x=2+h$ $y=(2+h)^2$. Therefore,

$$\text{av. grad.}=\frac{(2+h)^2-2^2}{(2+h)-2}=\frac{4h+h^2}{h}=4+h.$$

Example 2. Find the average gradient of the graph of $y=x^2$, (i) as x increases from -2 to $-1{\cdot}5$, (ii) as x increases from -2 to $-2+h$.

(i) In this case the increment of x is $-1{\cdot}5-(-2)$, that is, 0·5; the corresponding increment of y is $(-1{\cdot}5)^2-(-2)^2$, that is, $-1{\cdot}75$. Therefore,

$$\text{av. grad.}=\frac{-1{\cdot}75}{0{\cdot}5}=-3{\cdot}5.$$

(ii) The increment of x is $(-2+h)-(-2)$, that is, h; the corresponding increment of v is $(-2+h)^2-(-2)^2$, that is, $-4h+h^2$. Therefore,

$$\text{av. grad.}=\frac{-4h+h^2}{h}=-4+h.$$

Note that the increment is always calculated by subtracting the value *from which* the variable changes from the value *to which* it changes.

Example 3. What is the average rate at which the function $16t^2$ changes, (i) as t changes from 2 to $2+h$; (ii) as t changes from -2 to $-2+h$?

$$\text{(i) av. rate}=\frac{16(2+h)^2-16(2)^2}{(2+h)-2}=64+16h;$$

$$\text{(ii) av. rate}=\frac{16(-2+h)^2-16(-2)^2}{(-2+h)-(-2)}=-64+16h.$$

Let $16t^2$ be the number of feet a stone falls in t seconds; then the increment of $16t^2$, as t increases from 2 to $2+h$ (namely, $64h+16h^2$), is the distance in feet which the stone falls during the fraction h of a second succeeding the first two seconds of its fall. The quotient of this increment by the corresponding increment h of t measures the **average velocity** during the interval; the average velocity is therefore $64+16h$ feet per second.

8. Turning Points. Maxima and Minima. The simplest way of studying the properties of a function is often by examination of its graph. Suppose a point to start from A and to move along the curve $ABC...$ (Fig. 4), and con-

sider in the first place how the ordinate changes during the motion of the point; the variation of the function is, of course, represented by the variation of the ordinate.

As the point moves from A to B it rises, and the ordinate increases; when the point passes B it begins to descend, and the ordinate begins to decrease. The point B is called **a turning point** of the graph, and the ordinate at B, namely $B'B$, is called a **turning value** of the ordinate; by analogy, the value of the function represented by $B'B$ is called a turning value of the function.

As the point moves further along the curve it continues to descend till it reaches D, but on passing D it begins to ascend. D is therefore another turning point of the curve,

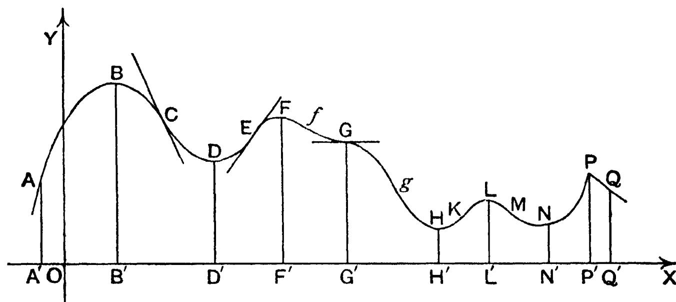


Fig. 4.

and $D'D$ a turning value of the ordinate, or of the function which the curve represents.

Other turning points are F, H, L, N, P.

The ordinate at B is greater than any other ordinate *near it and on either side of it*; the value of the ordinate at B is therefore said to be a **maximum**. At D, on the other hand, the ordinate is less than any other ordinate near it and on either side of it; the value of the ordinate at D is therefore said to be a **minimum**.

The ordinates at F, L, and P are maximum ordinates; those at H and N are minimum ordinates.

The meaning just given of the words "maximum" and "minimum" is that usually assigned to them in mathematics. A maximum ordinate is not necessarily, though it

sometimes is, the greatest ordinate of the curve; an ordinate is a maximum if it is greater than any other ordinate near it and on either side of it. A maximum may even be less than a minimum ordinate; thus $L'L$ is a maximum ordinate but it is less than the minimum ordinate $D'D$.

For the values of x corresponding to the arcs AB, DF, ... the function is said to be an **increasing function**; as x increases from OA' to OB', or from OD' to OF', the value of the function increases. On the other hand, for the values of x corresponding to the arcs BD, FH, ... the function is said to be a **decreasing function**; as x increases from OB' to OD', or from OF' to OH', the function decreases. A turning value of a function is a value at which the function either ceases to increase and begins to decrease, or else ceases to decrease and begins to increase.

9. Variation of the Gradient. Point of Inflexion. We shall now consider how the gradient of the curve varies, and in this discussion we shall think of the gradient at any point of the curve as the (trigonometrical) tangent of the angle that the tangent to the curve at the point makes with the x-axis, which we suppose to be horizontal.

Let us first make clear what is meant by the angle that a tangent makes with the x-axis. By the angle between a tangent and the x-axis we mean the *acute* angle between the tangent and the *positive direction* of the x-axis. If we suppose the line AB in Figs. 1, 2 to be the tangent to a curve at the point P then, in both figures, the angle that AB makes with the x-axis is the angle XCB or its equal, the angle RPQ. In Fig. 2, however, AB has a right-hand downward slope and the angle XCB is *negative*; the gradient, which is equal to $\tan XCB$, is also negative.

Another point to be attended to is this, that we are concerned not merely with numerical but with *algebraical* increase. When the angle XCB is negative so is $\tan XCB$; as XCB increases numerically, the gradient $\tan XCB$ *decreases algebraically*. In other words, when a line has a right-hand downward slope, the greater (numerically) the angle of the slope the *less* (algebraically) is the gradient of the line.

Let us now suppose that, as a point moves along the curve $AB\ldots$ (Fig. 4), the tangent is drawn to the curve at each position of the moving point, and let us notice how the gradient of the tangent changes. It will be convenient to think of a moving line accompanying the moving point; when the point is at A we shall suppose the line to coincide with the tangent at A and, as the point moves along the curve, we shall suppose the line to move with it in such a way as to coincide always with the tangent to the curve in each position of the moving point.

As the point moves from A to B the angle of slope is positive but decreasing, and therefore the gradient is positive and decreasing; the moving line rotates *clockwise.* At B the moving line is horizontal and the gradient is zero. As the point moves from B to C the angle of slope is negative and increasing *numerically,* but the gradient, being now negative, is still decreasing *algebraically*; the moving line continues to rotate clockwise.

When the point passes C, the angle of slope begins to decrease numerically and therefore (since the angle of slope is negative) the gradient begins to increase algebraically; the moving line now rotates *anticlockwise.* At C, therefore, the gradient ceases to decrease and begins to increase; **the gradient** has a **turning value** at C, and this value is a minimum.

As the point moves from C to D the angle of slope decreases numerically and the gradient increases algebraically; the moving line rotates anticlockwise. At D the moving line is horizontal and the gradient is zero. As the point moves from D to E, the angle of slope is now positive and increasing and the gradient, which is now also positive, continues to increase; the moving line also continues to rotate anticlockwise.

When the point passes E, however, the angle of slope and the gradient begin to *decrease* while the rotation of the moving line again becomes clockwise. At E therefore the gradient ceases to increase and begins to decrease; the gradient has a turning value at E, just as it has at C, but at E the turning value is a maximum.

Proceeding in this way, we see that turning values of the

gradient occur at f, G, g, K, M. At G the moving line is horizontal.

Definition. A point on a curve at which the gradient has a turning value is called a **point of inflexion**, and the tangent at a point of inflexion is called an **inflectional tangent.**

It will be noticed that at a point of inflexion the curve *crosses* the tangent; the character of the curve near a point of inflexion is shown more clearly in Fig. 5.

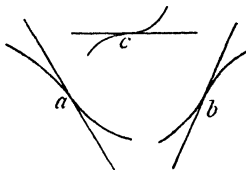


Fig. 5.

We may further note that the portion ABC of the graph (Fig. 4) is **convex upwards**, while the portion CDE is **concave upwards**. So long as the gradient is a decreasing function, the curve is convex upwards, while so long as the gradient is an increasing function the curve is concave upwards. The point where the change from convexity to concavity takes place is a point of inflexion.

10. Continuity. In tracing curves from their equations the student must have observed, that *near* their turning points the ordinate usually changes very slowly, and that *at* the turning points the gradient is zero. It is, however, quite possible to have a turning point near which the ordinate does not change slowly and at which the gradient is not zero. The point P (Fig. 4) is such a point.

From the graphical point of view, a curve which has a sharp peak like P is simple enough; but, as a matter of fact, no curve given by an ordinary equation such as we shall deal with ever shows such peaks. The occurrence of a peak is usually associated with a **discontinuity** of the gradient. We shall refer very briefly to the question of continuity.

In all ordinary functions (except, it may be, near a

limited number of values of the argument) a small change in the argument produces only a small change in the function. This property is reflected in the graph; the graph as a rule forms a continuous, uninterrupted line. Suppose, however, that we consider the variation of the *gradient* of Fig. 4. As a point moves along the curve the gradient changes steadily till the point approaches P; as it approaches P the gradient is positive, but when it passes through P the gradient suddenly becomes negative. The gradient passes from a positive to a negative value, *jumping over* (so to speak) a whole series of values. In mathematical language the gradient is said to be *discontinuous* at P.

The ordinate of the curve, on the other hand, is continuous all along the curve; in passing from any one value to any other value, from $A'A$ to $Q'Q$ say, it takes once at least every value between $A'A$ and $Q'Q$, and it makes no jump anywhere.

It will always be assumed that every variable is a continuous variable. This assumption implies (i) that a small increment of the argument of a function produces only a small increment of the function, and (ii) that as the variable changes, say from a to b, it assumes once at least every value between a and b.

11. Notation for Functions. A function of a variable is often denoted by enclosing the variable in a bracket and prefixing a letter; thus, $f(x)$, $F(x)$, $\phi(x)$ denote functions of x. The letters f, F, ϕ are functional symbols, not multipliers: the symbol $f(x)$ must be taken *as a whole* and means simply "some function of x," the context or an explicit statement determining which particular function is meant. For different functions occurring in the same investigation different functional symbols must, of course, be used.

$f(a)$ means "the value of the function $f(x)$ when x has the value a," or "the value of the function $f(x)$ when x is replaced by a." Thus, if $f(x)$ denotes the function

$$2x^2-3x-4,$$

then $$f(0)=-4,\ f(2)=-2,\ f(-2)=10,$$

$$f(x_1)=2x_1^2-3x_1-4,\ f(x_1+h)=2(x_1+h)^2-3(x_1+h)-4.$$

The beginner should note that an accent or a suffix attached to the functional letter indicates a *different function* from that denoted by the same letter without the accent or suffix; thus, $f(x)$, $f'(x)$, $f_1(x)$ denote three different functions. An accent or suffix is often used to indicate that, though the function whose letter bears the accent or suffix is a different function, yet it has some special connection with the function denoted by the same letter without an accent or suffix.

12. Notation for Increments. If P is the point (x_1, y_1) and Q the point (x_2, y_2) then, as a point passes along the arc PQ, the x of the point changes from x_1 to x_2, and the difference $x_2 - x_1$ (not $x_1 - x_2$) has been called (§ 6) the increment of x; similarly, $y_2 - y_1$ is the increment of y. We shall frequently have occasion to deal with increments, and it is convenient to have a notation for them. We may, if we please, denote $x_2 - x_1$ by a single letter, say h, and $y_2 - y_1$ by a single letter, say k; then

$$x_2 - x_1 = h, \;\; x_2 = x_1 + h; \;\; y_2 - y_1 = k, \;\; y_2 = y_1 + k.$$

The coordinates of Q (those of P being x_1, y_1) are now $x_1 + h$, $y_1 + k$, and the gradient of the chord PQ is k/h.

There is, however, a more suggestive notation for an increment, namely, the letter δ or Δ* prefixed to the value of x or y *from which the increment begins*: thus, δx_1 or Δx_1, δy_1 or Δy_1. The symbol δx_1, pronounced "delta x_1," must be taken *as a whole*; δ is not a multiplier. The square, cube, ... of δx_1 are written $(\delta x_1)^2$, $(\delta x_1)^3$, ...

In this notation the gradient of the chord PQ is $\delta y_1/\delta x_1$.

We shall now work some examples; after reading these, the student should try Exercises I. It is necessary for him to be quite familiar with the notation.

Example 1. If $f(x) = x^3 - 3x + 1$, find the value of $f(0)$, $f(1)$, $f(-1)$, and write down $f(a+h)$ in ascending powers of h.

$f(x)$ means the expression or function $x^3 - 3x + 1$; $f(0)$ means the value of $f(x)$ when $x = 0$. Hence

$$f(0) = 0 - 0 + 1 = 1.$$

* δ and Δ are the forms in the Greek alphabet of the small and capital d, the first letter of the word "difference."

Similarly, $f(1)=1-3\times 1+1=-1,$

and $f(-1)=(-1)^3-3\times(-1)+1=3.$

Again, to find $f(a+h)$ replace x in $f(x)$ by $a+h$; therefore

$$f(a+h)=(a+h)^3-3(a+h)+1$$
$$=a^3+3a^2h+3ah^2+h^3-3a-3h+1.$$

Arranging in ascending powers of h, we find

$$f(a+h)=a^3-3a+1+3(a^2-1)h+3ah^2+h^3.$$

The terms independent of h are simply $f(a)$; we may therefore write

$$f(a+h)=f(a)+3(a^2-1)h+3ah^2+h^3.$$

How could we show, without calculation, that the terms independent of h must be $f(a)$?

Example 2. If $F(x)=2x^2-3x-4$, find the gradient of the chord joining the points on the graph of $F(x)$ whose abscissae are 1 and 1·5.

$$F(1)=-5 \text{ and } F(1\cdot5)=-4.$$

Incr. of abscissa $=1\cdot5-1=0\cdot5$; incr. of ordinate $=-4-(-5)=1.$

Hence $\text{grad.}=\dfrac{\text{incr. of ord.}}{\text{incr. of abs.}}=\dfrac{1}{0\cdot5}=2.$

Example 3. If $F(x)=2x^2-3x-4$, find the gradient of the chord joining the points on the graph of $F(x)$ whose abscissae are x_1 and $x_1+\delta x_1$.

$$F(x_1)=2x_1^2-3x_1-4,$$
$$F(x_1+\delta x_1)=2(x_1+\delta x_1)^2-3(x_1+\delta x_1)-4$$
$$=2x_1^2-3x_1-4+(4x_1-3)\delta x_1+2(\delta x_1)^2$$
$$=F(x_1)+(4x_1-3)\delta x_1+2(\delta x_1)^2.$$

The increment of the ordinate, corresponding to the increment δx_1 of the abscissa, is $F(x_1+\delta x_1)-F(x_1)$ or, in the notation of increments, $\delta F(x_1)$. Now

$$\delta F(x_1)=F(x_1+\delta x_1)-F(x_1)=(4x_1-3)\delta x_1+2(\delta x_1)^2,$$

and therefore $\text{grad.}=\dfrac{\delta F(x_1)}{\delta x_1}=4x_1-3+2\delta x_1.$

Example 4. If in Example 3 we give to δx_1 in succession the values 1, 0·1, 0·01, what are the successive values of the gradient?

If P is the point whose abscissa is x_1, and Q the point whose abscissa is $x_1+\delta x_1$, the question is equivalent to this: what are the gradients of the three chords obtained by joining P to the three positions of Q? The third position of Q is very close to P, and therefore the gradient of PQ in this case must be very nearly equal to the gradient of the tangent at P.

To obtain the required values, we have merely to substitute 1, 0·1, 0·01 for δx_1 in the expression for the gradient. We find

$$4x_1-1,\ 4x_1-2\cdot8,\ 4x_1-2\cdot98.$$

Example 5. If $y=\log x$ find the value of $\delta y_1/\delta x_1$ when $x_1=20$ and $\delta x_1=2$, 1, 0·5, using four-figure logarithms.

$$\delta x_1=2,\quad \delta y_1=\log 22-\log 20,\quad \frac{\delta y_1}{\delta x_1}=\frac{0{\cdot}0414}{2}=0{\cdot}0207,$$

$$\delta x_1=1,\quad \delta y_1=\log 21-\log 20,\quad \frac{\delta y_1}{\delta x_1}=\frac{0{\cdot}0212}{1}=0{\cdot}0212,$$

$$\delta x_1=0{\cdot}5,\quad \delta y_1=\log 20{\cdot}5-\log 20,\quad \frac{\delta y_1}{\delta x_1}=\frac{0{\cdot}0108}{0{\cdot}5}=0{\cdot}0216.$$

When the value of x is large, say $x=500$, and the values of δx are small integers or proper fractions the four-figure tables are not sufficiently accurate to show the relative magnitudes of the gradients.

EXERCISES. I.

1. If $f(x)=5x^2-7x+2$, calculate $f(0), f(1), f(2), f(-1)$. Show that
$$f(x_1+\delta x_1)=f(x_1)+(10x_1-7)\delta x_1+5(\delta x_1)^2.$$

2. If $f(x)=x^3+x^2+x+1$, calculate $f(0)$, $f(\frac{1}{2})$, $f(-\frac{1}{2})$. Show that
$$f(x_1+\delta x_1)=f(x_1)+(3x_1{}^2+2x_1+1)\delta x_1+(3x_1+1)(\delta x_1)^2+(\delta x_1)^3.$$

3. If $F(x)=3x^2+2x-1$ write down $F(ax+b)$, $F(x^2)$, $F(x^3)$.

4. If $f(x)=\sin x$, the angle being measured in degrees, calculate
$$f(0),\ f(30),\ f(47{\cdot}5),\ f(90).$$

5. If $f(x)=3\sin x+4\cos x$, the angle being measured in radians, calculate $f(0), f(\pi/2), f(\pi), f(1), f(0{\cdot}5)$.

6. If $F(x)=\log x$ show that
$$F(x)+F(y)=F(x\times y),\quad F(x)-F(y)=F\left(\frac{x}{y}\right).$$

Calculate the gradient of the chord joining the points (x_1, y_1) and $(x_1+\delta x_1, y_1+\delta y_1)$ on each of the curves given by equations 7–14.

7. $y=3x$. **8.** $y=ax+b$. **9.** $y=ax^3+bx+c$.

10. $y=ax^3+bx^2+cx+d$. **11.** $y=1/x^2$.

12. $y=\log x$. **13.** $y=10^x$. **14.** $y=10^{-x}$.

15. Calculate $\delta y_1/\delta x_1$, when $y=\sin x$, for the following values of x_1 and δx_1; the angle is measured in radians.

(i) $x_1=0{\cdot}5236$; $\delta x_1=0{\cdot}0524$, $0{\cdot}0349$, $0{\cdot}0175$;

(ii) $x_1=2{\cdot}3090$; $\delta x_1=0{\cdot}0524$, $0{\cdot}0349$, $0{\cdot}0175$.

16. The same problem as in example 15 for $y=\cos x$.

17. The same problem as in example 15 for $y=\tan x$.

18. If $f(x)=a+bx^2+cx^4$ show that $f(-x)=f(x)$.

[When $f(-x)=f(x)$, the function $f(x)$ is said to be an **even function** of its argument. The simplest case of an even function is a rational, integral function which contains only even powers of its argument.]

19. If $f(x)=ax+bx^3+cx^5$ show that $f(-x)=-f(x)$.

[When $f(-x)=-f(x)$, the function $f(x)$ is said to be an **odd function** of its argument. The simplest case of an odd function is a rational, integral function which contains only odd powers of its argument.]

20. State which of the following functions are even, and which odd: $\sin x$, $\cos x$, $\tan x$, $\operatorname{cosec} x$, $\sec x$, $\cot x$, 10^x+10^{-x}, 10^x-10^{-x}, $(10^x-10^{-x})/x$.

CHAPTER II.

DIFFERENTIATION OF POWERS. MAXIMA AND MINIMA.

13. Tangent to a Curve. The ordinary conception of the tangent to a curve at a point P on it is probably this: the tangent is a straight line which meets the curve at P but which, if turned ever so little about P as a pivot, will again cut the curve near P. We shall put this conception in a slightly different form which will lead to a method of calculating the gradient of the tangent, and therefore of drawing the tangent itself.

Suppose the tangent PT at the point P (Fig. 6) to have been drawn, and consider its relation to any secant PS drawn through P and a neighbouring point Q on the curve. We do not assume that we know how to draw

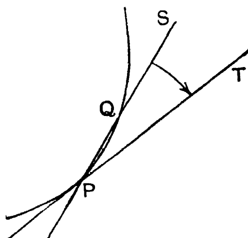


Fig. 6.

the tangent; we merely make a hypothetical construction for the sake of the argument. We can, however, always draw a secant; we merely have to take a point Q on the curve distinct from P and to join PQ.

Now, if Q is very near to P the angle TPQ will be very small, and the position of the secant PS will differ very

little from that of the tangent PT. Further, we can draw through P a secant that will make as small an angle as we please with PT; because we can take Q as close to P as we please and, so long as Q is distinct from P, no matter how close to it, we can draw the secant PS. (This last statement corresponds to the property of the tangent PT, that the rotation of PT through any angle, however small, will cause it to cut the curve again near P.)

The tangent PT is thus the line that *limits* the position of the secant PQ as Q approaches P—limits in this sense, that the angle TPQ becomes small as Q gets near to P and can be made as small as we please by taking Q close enough to P. Hence we may define a tangent as follows:

Definition. The tangent at a point P on a curve is a line PT such that the angle TPQ between PT and the secant PQ, through P and a neighbouring point Q on the curve, is small when Q is near to P and can be made as small as we please by taking Q near enough to P.

The following examples show the practical nature of the definition.

Example 1. Show how to draw the tangent at the point $P(2, 4)$ on the parabola $y=x^2$.

Let $OM=2$, $MP=4$; let Q be a point near P on the curve and draw PR parallel to the x-axis to meet the ordinate NQ at R (Fig. 7).

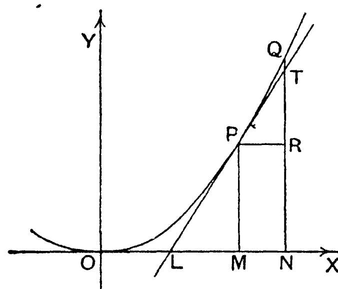


Fig. 7.

Calculate the gradient RQ/PR of the secant PQ; we have

$$NQ=ON^2=(2+MN)^2=4+4MN+MN^2,$$

and therefore
$$\frac{RQ}{PR}=\frac{NQ-MP}{MN}=4+MN.$$

Now, the closer Q is to P the less is the line MN; further, by taking N close enough to M, that is, by diminishing the length of MN far enough we can bring Q as close to P as we like. But the smaller MN becomes, the more nearly does the gradient RQ/PR become equal to 4.

Draw, then, through P the straight line PT whose gradient is 4. PT will be the tangent at P; because, the gradient of PT is 4, the gradient of PQ is $4+MN$, and we can take Q so close to P that the difference between these two gradients, namely MN, shall be as small as we please. In other words, we have found a line PT such that the angle TPQ can be made as small as we please by taking Q near enough to P; therefore PT is, by definition, the tangent at P.

Example 2. Find the gradient of the tangent at the point (x_1, y_1) on the parabola $y=x^2$.

In Fig. 7 let P be the point (x_1, y_1) and Q the point $(x_1+\delta x_1, y_1+\delta y_1)$; then the gradient of the secant PQ is given by the equation

$$\frac{\delta y_1}{\delta x_1}=\frac{(x_1+\delta x_1)^2-x_1^2}{\delta x_1}=2x_1+\delta x_1.$$

Now, when Q is close to P the increment δx_1, which is equal to MN, is small, and by making δx_1 small enough we can bring Q as close to P as we please. Let PT be the line through P whose gradient is $2x_1$.

PT is the tangent at P; because, the gradient of PT is $2x_1$, the gradient of PQ is $2x_1+\delta x_1$, and we can take Q so close to P that the difference between these two gradients, namely δx_1, shall be as small as we please. The gradient of the tangent at (x_1, y_1) is therefore $2x_1$.

14. Gradient of a Curve. By the gradient of a curve at a point P on it is meant (§ 7) the gradient of the tangent at P; it will, of course, except when the curve is a straight line, vary from point to point on the curve.* The definition of a tangent given in § 13 leads, as the examples worked in that article show, to a simple method of finding the gradient.

The process for obtaining the gradient of the tangent at P is the following:

First, we find the average gradient of the arc PQ (§ 7).

Secondly, we find the number to which the average gradient tends as the difference δx_1 between the abscissae of P and Q becomes smaller and smaller. In most cases, as in the examples of § 13, this number is evident when $\delta y_1/\delta x_1$ has been reduced to its simplest form.

* It may be well to warn the beginner that the word "curve" includes straight lines as well as curved lines.

As another example, find the gradient at the point P on the graph of $f(x)$, where $f(x)=x^3+x^2+x+1$, the abscissa of P being x_1.

In this case (Exercises I. 2)

$$f(x_1+\delta x_1)-f(x_1)=(3x_1{}^2+2x_1+1)\,\delta x_1+(3x_1+1)(\delta x_1)^2+(\delta x_1)^3,$$

and therefore the average gradient of the arc PQ is

$$\frac{\delta f(x_1)}{\delta x_1}=3x_1{}^2+2x_1+1+(3x_1+1)\delta x_1+(\delta x_1)^2.$$

The terms containing δx_1 and $(\delta x_1)^2$ can clearly be made as small as we please by making δx_1 sufficiently small. In other words, Q can be taken so near to P that the angle between PQ and the line through P whose gradient is $3x_1{}^2+2x_1+1$ shall be as small as we please. The gradient of the tangent at P is therefore $3x_1{}^2+2x_1+1$.

15. Limits. The process by which we pass from the gradient $\delta y_1/\delta x_1$ of the secant PQ to the gradient of the tangent at P is called **the method of limits.** When $\delta y_1/\delta x_1$ is equal to $2x_1+\delta x_1$, the gradient of the tangent, namely $2x_1$, is said to be "the limit of the gradient $2x_1+\delta x_1$ for δx_1 approaching zero as its limit."

It might seem at first sight as if we were merely taking a roundabout way of saying that $2x_1+\delta x_1$ is equal to $2x_1$ when δx_1 is equal to zero; but it is not so. Let us see how we found the average gradient $2x_1+\delta x_1$; we obtained it from the equation

$$\frac{\delta y_1}{\delta x_1}=\frac{2x_1\delta x_1+(\delta x_1)^2}{\delta x_1}=2x_1+\delta x_1. \qquad \text{.............. (1)}$$

Now, $2x_1+\delta x_1$ is obtained from the fraction by dividing its numerator and denominator by δx_1. This division is possible *provided* δx_1 *is not zero,* and not otherwise; division by zero is expressly excluded from the operations of algebra.* We cannot take a single step in our work if we suppose δx_1 to be zero; graphically, we should have only one point P and not two points P, Q through which to draw a straight line.

If the student reads § 13 over again he will see that we nowhere assume that Q ever coincides with P; rather, we

* It is surely a violation of common sense to perform a division on the express understanding that the divisor is not zero and then to assume that we are at liberty, after the division has been effected, to make the divisor zero. Such a proceeding is a mere juggling with symbols.

base our definition of a tangent on what seems to be a fair view of the ordinary conception of a tangent. PT will be the tangent at P provided we can show that the angle TPQ becomes as small as we please by taking Q sufficiently near to P.

In equation (1) we have a quantity δx_1 which we suppose to take smaller and smaller values, *tending towards zero*; in other words, δx_1 is a *variable* which tends to zero. The quantity $2x_1 + \delta x_1$ tends, as δx_1 tends to zero, to the definite value $2x_1$; this statement means simply that we can take δx_1 so small that $2x_1 + \delta x_1$ (and therefore also $\delta y_1/\delta x_1$) shall differ from $2x_1$ by as little as we please. The technical form of this last statement is "the limit of $2x_1 + \delta x_1$ is $2x_1$ when the limit of δx_1 is zero."

The variable which, in this process of finding the limit, acts as the independent variable, namely δx_1, may tend to some other number than zero as its limit. For example, the equation

$$\frac{x^2 - a^2}{x - a} = x + a$$

is an identity so long as x is different from a. If we suppose x equal to a the fraction takes the form 0/0, and this is a mere symbol without definite meaning. On the other hand, the fraction has a definite limit when x tends towards a as its limit. For, the equation has meaning and is true so long as x is different from a, and by taking x near enough to a we can make $x+a$ (and therefore also the fraction) as nearly equal to $2a$ as we please. The limit of the fraction, when x approaches a as its limit, is therefore $2a$.

16. Definition of a Limit. Notation. We shall now give a formal definition of a limit.

Definition. When it is possible to make the argument of a given function so nearly equal to a definite number a that the function will differ from another definite number A by as little as we please, that difference remaining as small as we please when the argument is taken still nearer to a, then A is called the limit of the function for the argument tending to (or converging to, or approaching) a as *its* limit.

The notation for a limit is the letter L, or the first three letters of the word limit, namely *lim*. The statement that a function y of x has A for its limit, when x has a for *its* limit, is represented thus:

$$\mathrm{L}y = A \quad \text{when} \quad \mathrm{L}x = a,$$

or, more usually,
$$\underset{x=a}{\mathrm{L}}\, y = A.$$

This last equation is read "the limit of y for x equal to a is A"; it must be remembered, however, that the phrase "the limit ... for x equal to a" is a mere contraction for "the limit ... for x converging to a as its limit."

In this notation we have, for example,

$$\underset{\delta x_1 = 0}{\mathrm{L}} \frac{(x_1+\delta x_1)^2 - x_1^2}{\delta x_1} = 2x_1; \quad \underset{x=a}{\mathrm{L}} \frac{x^2 - a^2}{x - a} = 2a.$$

It may happen that the argument becomes infinite, that is, becomes and remains greater than any assigned number N, no matter how large N may be. We thus have such cases as the following:

$$\underset{x=\infty}{\mathrm{L}} \frac{1}{x} = 0, \quad \underset{x=\infty}{\mathrm{L}} \tan^{-1} x = \frac{\pi}{2},$$

$$\underset{x=\infty}{\mathrm{L}} \frac{x^2+5x+7}{2x^2-8x+3} = \underset{x=\infty}{\mathrm{L}} \frac{1+\dfrac{5}{x}+\dfrac{7}{x^2}}{2-\dfrac{8}{x}+\dfrac{3}{x^2}} = \tfrac{1}{2}.$$

Practically, the only theorem for working with limits that we shall need is this: when the limit of δx_1 is zero the limit of $A \times \delta x_1$ is also zero, where A is finite and either does not contain δx_1 at all or, if it does contain δx_1, remains finite as δx_1 tends to zero.

The theorem hardly requires proof, but we may give a formal demonstration. To show that the limit of $A\delta x_1$ is zero when the limit of δx_1 is zero, we must show that we can take δx_1 so small (not equal to zero, but only so nearly equal to zero) that $A\delta x_1$ will be as small as we please. Suppose, for example, that we want to have $A\delta x_1$ less than $1/10^6$. If A itself varies with δx_1 let A' be its greatest numerical value for any of the values that δx_1 takes. Then, to make $A\delta x_1$ less than $1/10^6$ we need only take δx_1 less than $1/A'10^6$; this choice of δx_1 is possible because, since the limit of δx_1 is zero, we can take δx_1 as

small as we please. Note that unless A is zero we cannot make $A\delta x_1$ zero, because zero is not a value that δx_1 takes in the process.

Example 1. Show that

$$\underset{h=0}{\mathrm{L}} \frac{(x+h)^3 - x^3}{h} = 3x^2.$$

We have $\dfrac{(x+h)^3 - x^3}{h} = 3x^2 + 3hx + h^2 = 3x^2 + (3x+h)h.$

The term $(3x+h)h$ can be made as small as we please by diminishing h; the limit, when h converges to zero, is therefore $3x^2$.

Example 2. What is the value of $(x^2+x-2)/(2x^2+x-3)$ when $x=1$? When $x=1$ we find that

$$\frac{x^2+x-2}{2x^2+x-3} = \frac{1+1-2}{2+1-3} = \frac{0}{0}.$$

Nearly every beginner says that the symbol we have obtained for the fraction represents 1, probably because a fraction whose numerator and denominator are equal is equal to 1. But such a conclusion implies that the numerator and denominator are not zero. What then *is* the value of the fraction when $x=1$? The correct answer is that the fraction has not any value; this does not mean that the fraction *has the value which is called* "*nothing*," but that its value is *not defined*. The symbol 0/0 has in itself no meaning whatsoever.

The fraction has, however, a definite *limit* when x converges to 1 as its limit. For, both numerator and denominator contain the factor $x-1$ and, so long as x is different from 1, we may divide them both by $x-1$. Hence, if x is not equal to 1,

$$\frac{x^2+x-2}{2x^2+x-3} = \frac{(x-1)(x+2)}{(x-1)(2x+3)} = \frac{x+2}{2x+3}.$$

The fraction $(x+2)/(2x+3)$ is equal to 3/5 when $x=1$, and can be made as nearly equal as we please to 3/5 by making x sufficiently nearly equal to 1; therefore its *limit* for x *converging to* 1 is 3/5, the same as its *value* when x is *equal to* 1. The given fraction is equal to $(x+2)/(2x+3)$ so long as x is not equal to 1; therefore the given fraction may be made as nearly equal as we please to 3/5, by taking x sufficiently nearly equal to 1. In other words, the limit of the given fraction is 3/5.

17. Gradient of the Graph of Powers of x. We shall now find the gradient when y or $f(x)$ is of the form

$$ax^n + bx^{n-1} + cx^{n-2} + \dots + kx + l. \quad\text{..............(1)}$$

The abscissa of the point P at which the gradient is calculated is x_1 and that of the point Q, near to P, is $x_1 + \delta x_1$.

(i) Take first the case $y=f(x)=x^3$. The average gradient of the arc PQ is

$$\frac{\delta y_1}{\delta x_1}=\frac{f(x_1+\delta x_1)-f(x_1)}{\delta x_1}=\frac{(x_1+\delta x_1)^3-x_1^3}{\delta x_1}.$$

When the fraction is reduced we find

$$\frac{\delta y_1}{\delta x_1}=3x_1^2+3x_1\delta x_1+(\delta x_1)^2=3x_1^2+(3x_1+\delta x_1)\delta x_1,$$

and therefore the gradient at P is

$$\underset{\delta x_1=0}{\mathrm{L}}\frac{\delta y_1}{\delta x_1}=\underset{\delta x_1=0}{\mathrm{L}}\{3x_1^2+(3x_1+\delta x_1)\delta x_1\}=3x_1^2. \quad\text{......(2)}$$

(ii) Let $y=f(x)=x^n$, where n is a positive integer.

Here $$f(x_1+\delta x_1)=(x_1+\delta x_1)^n$$
$$=x_1^n+nx_1^{n-1}\delta x_1+A(\delta x_1)^2; \quad\text{..........(3)}$$

where A contains x_1 and δx_1 and their powers; the third term of the binomial expansion contains $(\delta x_1)^2$, the fourth term contains $(\delta x_1)^3$ and so on, and therefore every term after the second contains the second power of δx_1. We now have

$$\frac{\delta y_1}{\delta x_1}=\frac{(x_1+\delta x_1)^n-x_1^n}{\delta x_1}=nx_1^{n-1}+A\delta x_1, \quad\text{...........(4)}$$

and therefore the gradient at P is

$$\underset{\delta x_1=0}{\mathrm{L}}\frac{\delta y_1}{\delta x_1}=\underset{\delta x_1=0}{\mathrm{L}}(nx_1^{n-1}+A\delta x_1)=nx_1^{n-1}. \quad\text{..........(5)}$$

We will assume that this result holds also when n is fractional or negative; a complete proof will be found in the author's *Calculus*, § 57.

(iii) Let $y=f(x)=ax^3+bx^2+cx+d$. In this case

$$f(x_1+\delta x_1)=a(x_1+\delta x_1)^3+b(x_1+\delta x_1)^2+c(x_1+\delta x_1)+d,$$
$$f(x_1+\delta x_1)-f(x_1)=(3ax_1^2+2bx_1+c)\delta x_1$$
$$+(3ax_1+b)(\delta x_1)^2+a(\delta x_1)^3,$$
$$\frac{\delta y_1}{\delta x_1}=3ax_1^2+2bx_1+c+(3ax_1+b+a\delta x_1)\delta x_1, \quad\text{..(6)}$$

and therefore the gradient at P is

$$\underset{\delta x_1=0}{\mathrm{L}}\frac{\delta y_1}{\delta x_1}=3ax_1^2+2bx_1+c. \quad\text{....................(7)}$$

When there are more terms the process and results are similar.

Notice that the constant *term* d does not appear in the gradient, but that the constant *factors* a, b, c remain as factors. The geometrical reason for the absence of d from the gradient is obvious.

The reasoning is the same whatever particular value of x, such as x_1, we take. If we use the phrase "gradient of the function" instead of "gradient of the graph" we may state the results we have obtained as follows:

$$\text{grad. of } x^3 = 3x^2;\ \text{grad. of } x^n = nx^{n-1}, \quad \text{......... (A)}$$

$$\text{grad. of } ax^3 + bx^2 + cx + d = 3ax^2 + 2bx + c; \quad \text{...... (B)}$$

and for the function (1)

$$\text{grad. of } ax^n + bx^{n-1} + cx^{n-2} + \ldots + kx + l$$
$$= nax^{n-1} + (n-1)bx^{n-2} + (n-2)cx^{n-3} + \ldots + k. \quad \text{... (C)}$$

Example. Write down the gradients in the following cases:

(i) x^4, (ii) $3x^3 - 4x^2 + 5x + 7$, (iii) $3x^{\frac{1}{2}} + 5x^{-\frac{1}{2}}$.

The results are:

(i) grad. of $x^4 = 4x^3$;

(ii) grad. of $3x^3 - 4x^2 + 5x + 7 = 9x^2 - 8x + 5$;

(iii) grad. of $3x^{\frac{1}{2}} + 5x^{-\frac{1}{2}} = \frac{1}{2} \times 3x^{\frac{1}{2}-1} + (-\frac{1}{2}) \times 5x^{-\frac{1}{2}-1}$
$= \frac{3}{2}x^{-\frac{1}{2}} - \frac{5}{2}x^{-\frac{3}{2}}$.

18. Derivatives. Differentiation. The gradient is itself a function of the argument x and it has received various names; as, "the derivative of y or $f(x)$ with respect to x," "the differential coefficient of y with respect to x," "the derived function of y with respect to x." The phrase "x-derivative" may be used instead of "derivative with respect to x."

Of course when the argument is not x but some other variable, say t or u, the derivative is a t-derivative or a u-derivative. The argument need not be named when it is quite clear what it is.

The process of finding the derivative is called **differentiation**; to differentiate a function means to find its derivative.

There are several notations for the derivative. A very simple one is a capital D (the first letter of the word)

prefixed to the function; thus, Dy, $Df(x)$, $D(3x^2+2x-1)$. When the argument has to be specified it is placed as a suffix to D; thus, D_xy, D_ty.

A more common notation is that which represents the derivative as a fraction; thus,

$$\frac{dy}{dx}, \quad \frac{df(x)}{dx}, \quad \frac{d(3x^2+2x-1)}{dx}.$$

This notation is suggested by the quotient of the increments, namely $\frac{\delta y}{\delta x}$. The symbol $\frac{dy}{dx}$ must for the present be taken *as a whole;* we shall afterwards (§ 22) assign a meaning to the symbols dy and dx. When the function consists of several terms the derivative is sometimes written thus:

$$\frac{d}{dx}(3x^2+2x-1).$$

In this form the symbol $\frac{d}{dx}$ (which must be taken *as a whole,* just as the form dy/dx must be taken as a whole) is equivalent to the other symbol D_x.

Lastly, when the function is represented by the symbol $f(x)$, or $F(x)$, the derivative is often indicated by putting an accent over the functional letter; thus, $f'(x)$, $F'(x)$.

To indicate the value of a derivative for a particular value of x, the following notations are used:

$$[Dy]_{x=a}; \quad \left[\frac{dy}{dx}\right]_{x=a}; \quad f'(a).$$

The last of these is specially convenient.

The diversity of names and notations is apt to perplex the beginner, but they are all in use and have their own advantages. It is best therefore to master them at once; the difficulty is after all more apparent than real.

We shall now work some examples, but we note specially the following results, proved in § 17.

I. To find the derivative of x^n **multiply** by the index n and then **subtract** 1 from the index n (§ 17, A).

II. The derivative of the sum of two or more terms is equal to the sum of the derivatives of the terms (§ 17, B, C).

III. A constant **term** in the function does not appear in the derivative. We may say that **the derivative of a constant is zero.**

IV. A constant **factor** of a term remains as a factor of the corresponding term in the derivative (§ 17, B, C).

The following result is often useful:

V. $D(ax+b)^n = na(ax+b)^{n-1}$. In words, to find the derivative of the n^{th} power of the **linear function** $ax+b$, multiply by the index n **and by the coefficient** a and then subtract 1 from the index n (see example 5 below).

Example 1. Find the derivative when $y=f(x)=4x^2-8x-7$.

$$Dy=D(4x^2-8x-7)=8x-8\text{ ; or, }f'(x)=8x-8.$$

The gradients of the graph for the values 0, 1, 2, -1 of x are given by the equations

$$f'(0)=-8,\ f'(1)=0,\ f'(2)=8,\ f'(-1)=-16.$$

When $x=0$, the tangent has a right-hand downward slope of 8 in 1; when $x=1$, the tangent is horizontal; when $x=2$, it has a right-hand upward slope of 8 in 1; when $x=-1$, it has a right-hand downward slope of 16 in 1. (Compare Fig. 8.)

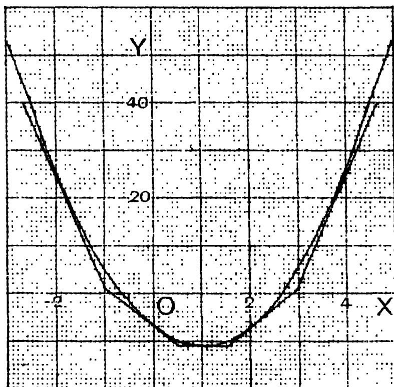


Fig. 8.

Note. To draw the tangent at P when its gradient is -8, move to the right one *horizontal* unit, then downwards 8 *vertical* units; or, move to the right one-tenth of the

horizontal unit; then downwards eight-tenths of the vertical unit, etc., and join P to the point so found. In drawing the tangent attention to the scales of the figure is necessary.

Example 2. Differentiate $y=\sqrt{x^3}+\frac{2}{x}-\frac{3}{x^2}+\frac{4}{\sqrt{x^5}}$.

Write $$y=x^{\frac{3}{2}}+2x^{-1}-3x^{-2}+4x^{-\frac{5}{2}};$$

then $$\frac{dy}{dx}=\tfrac{3}{2}x^{\frac{1}{2}}+(-1)2x^{-2}-(-2)3x^{-3}+(-\tfrac{5}{2})4x^{-\frac{7}{2}}$$
$$=\tfrac{3}{2}x^{\frac{1}{2}}-2\frac{1}{x^2}+6\frac{1}{x^3}-10\frac{1}{x^{\frac{7}{2}}}$$
$$=\tfrac{3}{2}\sqrt{x}-\frac{2}{x^2}+\frac{6}{x^3}-\frac{10}{\sqrt{x^7}}.$$

The student should make sure that he knows the working rules of indices. A common blunder is to *add* 1 when the index is negative, making, for example, Dx^{-2} equal to $-2x^{-1}$ instead of $-2x^{-3}$.

A frequently occurring result is that of the derivative of $\sqrt{x}$;
$$D(\sqrt{x})=Dx^{\frac{1}{2}}=\tfrac{1}{2}x^{-\frac{1}{2}}=\frac{1}{2\sqrt{x}}.$$

Example 3. Find dp/dv when (i) $pv=C$, (ii) $pv^{\gamma}=C$.
$$\text{(i)}\ p=\frac{C}{v}=Cv^{-1};\quad \frac{dp}{dv}=(-1)Cv^{-2}=-\frac{C}{v^2};$$
$$\text{(ii)}\ p=\frac{C}{v^{\gamma}}=Cv^{-\gamma};\quad \frac{dp}{dv}=-\gamma Cv^{-\gamma-1}=-\frac{\gamma C}{v^{\gamma+1}}.$$

Example 4. If $s=Vt-\frac{1}{2}gt^2$, find ds/dt.
$$\frac{ds}{dt}=V-gt.$$

Example 5. If $y=(ax+b)^n$, find dy/dx.

Here $$y_1+\delta y_1=\{a(x_1+\delta x_1)+b\}^n=\{(ax_1+b)+a\delta x_1\}^n,$$
so that $$y_1+\delta y_1=(ax_1+b)^n+n(ax_1+b)^{n-1}a\delta x_1+A(\delta x_1)^2,$$
and therefore, exactly as in § 17 (4),
$$\frac{\delta y_1}{\delta x_1}=na(ax_1+b)^{n-1}+A\delta x_1.$$
Hence $$\frac{dy}{dx}=\frac{d(ax+b)^n}{dx}=na(ax+b)^{n-1}.$$

The result of this example is very useful; it holds for all values of n. As particular cases we have, for instance,
$$D(3x+2)^2=2\times 3(3x+2)=18x+12.$$
$$D\sqrt{(4x-7)}=D(4x-7)^{\frac{1}{2}}=\tfrac{1}{2}\times 4(4x-7)^{-\frac{1}{2}}=\frac{2}{\sqrt{(4x-7)}}.$$

We can verify the first of these by expanding the square; thus,

$$(3x+2)^2=9x^2+12x+4\;;\;\; D(9x^2+12x+4)=18x+12.$$

Example 6. Find a function of x whose derivative is x^2-x+1.

To solve this little problem, we must remember the rules of differentiation. If *after* differentiation, the index of x is 2, then *before* differentiation it must have been 3; but in differentiating x^3 we multiply by 3, and therefore we must put in the factor $\frac{1}{3}$. Thus to get x^2, we differentiate $\frac{1}{3}x^3$. Similarly to get x, we differentiate $\frac{1}{2}x^2$, and to get 1 we differentiate x. Hence

$$\tfrac{1}{3}x^3-\tfrac{1}{2}x^2+x$$

is a function whose derivative is x^2-x+1, as may at once be tested by differentiation; this function is called an **integral** of x^2-x+1 and the process of finding it is called **integration.** Integration is taken up in Chapter V., but the student will find it to be good practice in differentiation to go through the inverse process of integration.

If any *constant* term, C say, be added to the function found above, this new function will still have x^2-x+1 as its derivative (§18, III.). The function containing this constant is called the **general integral.** For the present, the constant term need not be added to the integral.

Example 7. Write down the integral of (i) $\frac{1}{x^2}$, (ii) $\sqrt{x}$.

We find (i) Integral of $\frac{1}{x^2}=$ Int. of $x^{-2}=-x^{-1}=-\frac{1}{x}$.

(ii) Integral of $\sqrt{x}=$ Int. of $x^{\frac{1}{2}}=\frac{2}{3}x^{\frac{3}{2}}=\frac{2}{3}\sqrt{x^3}$.

Test the results by differentiating the functions obtained; thus,

$$D\left(-\frac{1}{x}\right)=D(-x^{-1})=-(-1)x^{-2}=x^{-2}=\frac{1}{x^2}.$$

EXERCISES. II.

Write down the derivatives of the functions of x in examples 1–30.

1. $5x$.	**2.** $5x+7$.	**3.** $4x^2$.
4. $4x^2-10$.	**5.** $7x^2-12x+6$.	**6.** $3+11x-2x^2$.
7. $3x^3-4x^2-x+1$.	**8.** $3-2x^2+4x^3-5x^4$.	**9.** $(x+1)(x+2)$.
10. $(3x-2)(2x+3)$.	**11.** $(x+1)(x+2)(x+3)$.	**12.** $(ax+b)(cx+f)$.
13. $x+\frac{1}{x}$.	**14.** $x^2+\frac{1}{x^2}$.	**15.** $4x+3-\frac{5}{x}$.
16. $ax+b+\frac{c}{x}$.	**17.** $\frac{3x^3+2x^2+1}{x}$.	**18.** $(2x+1)^3$.
19. $(2x+1)^5$.	**20.** $(1-x)^3$.	**21.** $(3-x)^4$.
22. $\frac{1}{x+1}$.	**23.** $\frac{1}{1-x}$.	**24.** $\frac{1}{(2x+3)^2}$.

25. $\frac{1}{(3-2x)^2}$. 26. $\sqrt{(2x+3)}$. 27. $\frac{1}{\sqrt{(2x+3)}}$.

28. $\sqrt[3]{(2-3x)}$. 29. $\frac{x^3-4x^2+5}{x+1}$. 30. $\frac{x^3-4x^2+5x+7}{(x-1)^2}$.

Differentiate with respect to the variables t, u, v the functions 31–40.

31. $10t-3t^2$. 32. $\frac{720}{v}$. 33. $\frac{720}{v^{\frac{3}{2}}}$. 34. $\frac{4}{v+3}$.

35. $a+bt+ct^2$. 36. $\sqrt{(3u-4)}$. 37. $\frac{1}{\sqrt{(3u-4)}}$.

38. $3u(u-5)$. 39. $at+b+\frac{c}{t}$. 40. $\frac{c}{v^{1 \cdot 4}}$.

Integrate with respect to x the functions 41–50, testing the results in each case by differentiation.

41. $3x^2+x-1$. 42. $x+1-\frac{1}{x^2}$. 43. $2x^2+\frac{1}{2x^2}$. 44. $\frac{1}{\sqrt{x}}$.

45. $\frac{2}{\sqrt{x}}-\frac{3}{\sqrt{x^3}}+\frac{6}{x^3}$. 46. $(x+1)^2$. 47. $(3x+2)^2$.

48. $\frac{1}{(3x+2)^2}$. 49. $\frac{1}{\sqrt{(3x+2)}}$. 50. $\frac{1}{4x^2-12x+9}$.

19. Rates. Variation of a Function. Before reading this article the student should glance again over § 8; the substance of that article will now be presented in a more definite form: the results obtained from inspection of the figure will be expressed in terms of numbers.

When x increases from x_1 to $x_1+\delta x_1$, the function $f(x)$ changes from $f(x_1)$ to $f(x_1+\delta x_1)$; the change in the function is therefore

$$f(x_1+\delta x_1)-f(x_1), \text{ that is, } \delta f(x_1).$$

The average rate at which $f(x)$ changes, as x increases from x_1 to $x_1+\delta x_1$, is $\delta f(x_1)/\delta x_1$. When $f(x)$ is represented by a graph this average rate is the gradient of the chord PQ of the graph, where P is the point on the graph whose abscissa is x_1 and Q the point whose abscissa is $x_1+\delta x_1$. (See § 7.)

The gradient of the tangent at P, that is, $f'(x_1)$ is the number which measures **the rate at which $f(x)$ is changing when x is equal to x_1.** (§ 7.)

In order therefore to find out *how* a function $f(x)$ changes as x increases from one value, a say, to another value b, we have merely to examine the value of its derivative $f'(x)$ for the values of x from $x=a$ to $x=b$. We go on to show that the mere sign of $f'(x)$ gives much information.

When $f'(x)$ is positive, the tangent has a right-hand upward slope; as the graphic point P moves to the right it also rises, and the value of the ordinate or function increases algebraically. (If the point P is below the x-axis a rise of P means that its ordinate decreases numerically but, since the ordinate is negative, this means that it increases algebraically.)

When $f'(x)$ is negative, the tangent has a right-hand downward slope; as P moves to the right it also falls, and the value of the ordinate or function decreases algebraically.

Hence, **the sign of the derivative $f'(x)$ tells whether the function $f(x)$ is increasing or decreasing.** To test *how* $f(x)$ changes as x increases from any value, a say, we first calculate $f'(a)$. If $f'(a)$ is a positive number, then $f(x)$ increases when x becomes greater than a; but if $f'(a)$ is a negative number, then $f(x)$ decreases when x becomes greater than a. If we suppose x to become less than a, then $f(x)$ will decrease when $f'(a)$ is positive, but will increase when $f'(a)$ is negative.

We have still to consider the case for which $f'(x)$ is zero. When $f'(x)=0$, the tangent is horizontal; the rate at which P is rising or falling, for those values of x that make $f'(x)$ vanish, is zero, and P is for the moment **stationary**. The value of the function is also said to be stationary; in most cases a stationary value is a turning value.

Hence, **to find the turning values of a function $f(x)$ we seek those values of x that make $f'(x)$ equal to zero.** For the method of deciding whether the turning value is a maximum or a minimum, see the following examples and § 21.

Example 1. If $f(x)=4x^2-8x-7$, trace the variation of the function as x increases from $-\infty$ to $+\infty$.

We have $$f'(x)=8(x-1).$$

If $x<1$, then $8(x-1)$ is negative. Therefore for every value of x less than 1 the derivative is negative, so that as x increases from $-\infty$ to 1 the function steadily decreases; $f(x)$ decreases from $+\infty$ to -11.

If $x > 1$, then $8(x-1)$ is positive. Therefore for every value of x greater than 1 the derivative is positive, so that as x increases from 1 to $+\infty$ the function steadily increases; $f(x)$ increases from -11 to $+\infty$.

If $x=1$, the derivative is zero. Therefore $f(1)$ is a stationary value of the function, obviously a minimum value.

These conclusions agree with the character of the graph of the function (Fig. 8, p. 29).

Example 2. Graph the function $f(x)$ where

$$f(x)=2x^3-9x^2-12x+24. \quad\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Find $f'(x)$ and express it in factors; we have

$$f'(x)=6x^2-18x-12=6\left\{(x-\tfrac{3}{2})^2-\left(\frac{\sqrt{17}}{2}\right)^2\right\},$$

or, since $\sqrt{17}=4{\cdot}123$ approximately,

$$f'(x)=6(x+0{\cdot}56)(x-3{\cdot}56). \quad\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

The derivative $f'(x)$ is zero when $x=-0{\cdot}56$ and when $x=3{\cdot}56$; the corresponding values of $f(x)$, namely $f(-0{\cdot}56)$ and $f(3{\cdot}56)$ where

$$f(-0{\cdot}56)=27{\cdot}55,\ f(3{\cdot}56)=-42{\cdot}55,$$

are stationary values of the function.

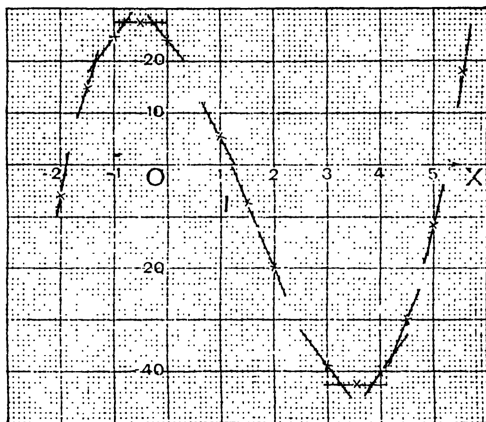


Fig. 9.

We must now examine the *sign* of $f'(x)$ as x varies from $-\infty$ to $+\infty$. Since a product changes its sign when *one of its factors* changes sign, we see from (2) that $f'(x)$ can only change sign when x passes through the values $-0{\cdot}56$ and $3{\cdot}56$; these two values of x guide us in tracing the sign of $f'(x)$.

If $x < -0·56$, that is, if x is negative and *numerically* greater than 0·56, both factors of $f'(x)$ are negative and therefore $f'(x)$ is positive. Hence as x increases from $-\infty$ to $-0·56$ the function $f(x)$ steadily increases and the tracing point of the graph steadily rises; since $f(-\infty) = -\infty$, the tracing point rises from the extreme low left to the point $(-0·56, 27·55)$.

If $x > -0·56$ but $< 3·56$, the first factor of $f'(x)$ is positive, the second factor is negative and therefore $f'(x)$ is negative. Hence as x increases from $-0·56$ to $3·56$ the function $f(x)$ decreases from 27·55 to $-42·55$. The stationary value $f(-0·56)$ is therefore a *maximum* value.

If $x > 3·56$, both factors of $f'(x)$ are positive and therefore $f'(x)$ is positive. Hence as x increases from 3·56 to $+\infty$ the function $f(x)$ increases from $-42·55$ to $+\infty$. The stationary value $f(3·56)$ is a *minimum* value.

We now know *how* $f(x)$ varies through its whole course. To show the use of the gradient in actually plotting the graph, we have marked in Fig. 9 certain points and have drawn short lengths of the tangents at these points. To complete the graph within the range indicated, we must join the points by a smooth curve and see that this curve touches the line at each point. It will be observed that the curve practically coincides with the tangent for a considerable distance on each side of the point of contact. The following values are shown:

x	-2	$-1·5$	-1	$-0·56$	0	1	1·5	2	3	3·56	4	4·5	5	5·5
$f(x)$	-4	15	25	27·55	24	5	$-7·5$	-20	-39	$-42·55$	-40	-30	-11	18·5
$f'(x)$	48	28·5	12	0	-12	-24	$-25·5$	-24	-12	0	12	28·5	48	70·5

Example 3. Find the point of inflexion on the graph of the function discussed in example 2.

A point of inflexion on a curve is one at which the gradient has a turning value (§ 9). Now, if the tabulated values of $f'(x)$ are examined it will be seen that they decrease from 48, when $x = -2$, to $-25·5$, when $x = 1·5$, and then increase to 70·5 when $x = 5·5$. The gradient has therefore a turning value at or near the point for which $x = 1·5$.

To test the matter definitely, let us for the moment choose a new functional symbol for the gradient, say $G(x)$; then

$$G(x) = f'(x) = 6x^2 - 18x - 12.$$

The turning value of the function $G(x)$ is obtained by finding the value of x for which the derivative $G'(x)$ vanishes; but

$$G'(x) = D(6x^2 - 18x - 12) = 12x - 18,$$

and therefore $G'(x) = 0$ when $x = 18/12 = 1·5$. When $x = 1·5$ the gradient $G(x)$ is a minimum; because $G'(x)$ is negative when $x < 1·5$

and positive when $x > 1{\cdot}5$. In other words, the gradient is a decreasing function so long as x is less than 1·5, and an increasing function so long as x is greater than 1·5; the value of the gradient when x is equal to 1·5 must therefore be a minimum.

We thus see that the point for which $x=1{\cdot}5$, that is, the point $(1{\cdot}5, -7{\cdot}5)$ is a point of inflexion on the graph of $f(x)$; but we have obtained the additional information that the graph has no other point of inflexion, because the gradient has only one turning value.

20. Derived Curve. In the last example we have used a new functional symbol for the gradient $f'(x)$. The gradient is a function of x, and we shall find it convenient for the present to use a special functional letter, such as G, to denote the gradient.

In plotting the graph of $f(x)$ it is convenient to employ a single letter, y say, to represent the ordinate. We may similarly employ a single letter, z say, to represent the ordinate of the *graph of* $G(x)$; the expression for the gradient being known, we may plot it just as we plot any other function of x.

Definition. The graph of the derivative of a function $f(x)$ of x is called **the derived curve** of the function $f(x)$.

We thus obtain for any function $f(x)$ two curves; one is the graph of the function itself and the other is the graph of the derivative of the function. In plotting the two curves the x-scale should be taken the same for both; the scale for the ordinates will usually need to be different. The ordinate of the derived curve will be proportional to the gradient at the corresponding point of the graph of the function.

For example 2, § 19, we shall have the equations

$$y=f(x)=2x^3-9x^2-12x+24, \qquad \text{(1)}$$

$$z=G(x)=f'(x)=6x^2-18x-12. \qquad \text{(2)}$$

The turning point of the graph of (2) is the point of inflexion of the graph of (1).

The graph of (2) will show clearly the various changes in the *gradient* of the graph of (1); it will in fact represent the variation described in § 9. When the ordinate z is positive, the ordinate y is *increasing*; when z is negative, y is *decreasing*; when z is zero, y is *stationary*.

This graphical method of tracing the variation of the gradient is often useful.

21. Maxima and Minima. The values of x that make $f(x)$ a maximum or a minimum are (§ 19) the roots of the equation $f'(x)=0$; but it may quite well happen that a value of x which makes $f'(x)$ vanish does not make $f(x)$ a maximum or a minimum. For example, if $f(x)=x^3$, then $f'(x)=3x^2$ and vanishes when $x=0$; but $f(x)$ is neither a maximum nor a minimum when $x=0$. The origin is a point of inflexion.

It is often troublesome to test whether a root of the equation $f'(x)=0$ does actually make $f(x)$ a maximum or a minimum. A straightforward, though often tedious, method is the following: let a be a root of $f'(x)=0$ and h a small positive number; calculate $f(a), f(a-h), f(a+h)$. The values found will show at once the nature of $f(a)$.

The best method is to examine the *signs* of $f'(a-h)$ and $f'(a+h)$. If $f'(a-h)$ is positive and $f'(a+h)$ negative, then $f(a)$ is a maximum; if $f'(a-h)$ is negative and $f'(a+h)$ positive, then $f(a)$ is a minimum. (See also §24, example 2.)

In many cases, however, the nature of the problem shows whether a maximum or a minimum exists, and then the value of x that makes $f'(x)$ vanish will give the solution.

Example 1. If $f(x)=4x^3-27x+5$, find the turning values.

$$f'(x)=12x^2-27=12(x+1·5)(x-1·5)$$

and $$f'(x)=0 \text{ when } x=-1·5 \text{ and when } x=+1·5.$$

We now test these values.

First Method. Take $h=0·5$; then

$f(-1·5)=32$; $f(-1·5-0·5)=f(-2)=27$; $f(-1·5+0·5)=f(-1)=28$, and therefore $f(-1·5)$ or 32 is a *maximum*;

$f(1·5)=-22$; $f(1·5-0·5)=f(1)=-18$; $f(1·5+0·5)=f(2)=-17$, and therefore $f(1·5)=-22$ is a *minimum*; $(-22$ is *algebraically* less than either -18 or $-17)$.

Second Method. It is only the *sign* of $f'(x)$ that we need; there is no necessity for calculating the value. We suppose h to be *small and positive.*

$$f'(-1·5-h)=12(-h)(-3-h)=12(-)(-)=+,$$
$$f'(-1·5+h)=12(h)(-3+h)=12(+)(-)=-.$$

Hence as x changes from $-1·5-h$ to $-1·5+h$, the derivative changes from + to −, and therefore $f(x)$ changes from an increasing to a decreasing function. $f(-1·5)$ is therefore a maximum value of $f(x)$.

Again,
$$f'(1{\cdot}5-h)=12(3-h)(-h)=12(+)(-)=-,$$
$$f'(1{\cdot}5+h)=12(3+h)(h)=12(+)(+)=+.$$

In this case $f'(x)$ changes from $-$ to $+$, and therefore $f(1{\cdot}5)$ is a minimum value of $f(x)$.

Note. The value $-1{\cdot}5$ of x *makes* $f(x)$ a maximum, or *gives* a maximum value of $f(x)$; it is $f(-1{\cdot}5)$ (that is, 32), that is the maximum, *not* $-1{\cdot}5$ as the beginner often *says*.

Example 2. If $f(x)=3x^4-8x^3+6x^2-10$ find the turning values.
$$f'(x)=12x^3-24x^2+12x=12x(x-1)^2,$$
so that $\qquad f'(x)=0$ when $x=0$ and when $x=1$.

Now, $\qquad f'(-h)=12(-h)(-h-1)^2=12(-)(+)=-,$
$$f'(h)=12(h)(h-1)^2=12(+)(+)=+.$$

Hence $f(0)$, which is equal to -10, is a *minimum*.

Again, $\qquad f'(1-h)=12(1-h)(-h)^2=12(+)(+)=+,$
$$f'(1+h)=12(1+h)(h)^2=12(+)(+)=+.$$

In this case $f'(x)$ *does not change sign* as x increases through the value 1; $f(x)$ *increases* as x increases from $1-h$ to 1 and *continues to increase* as x increases from 1 to $1+h$. The point on the graph of $f(x)$ for which $x=1$, namely $(1, -9)$, is a point of inflexion. We can confirm this by finding the points of inflexion on the graph of $f(x)$

Let $G(x)$ be the gradient; then $G(x)=f'(x)=12x^3-24x^2+12x$.
$$G'(x)=36x^2-48x+12=36(x-\tfrac{1}{3})(x-1).$$

When $x=\frac{1}{3}$, $G(x)$ is a maximum, and when $x=1$, $G(x)$ is a minimum.

Hence the points $(\frac{1}{3}, -\frac{259}{27}), (1, -9)$ are points of inflexion on the graph of $f(x)$.

Example 3. Find the greatest (right, circular) cone that can be inscribed in a sphere of radius R.

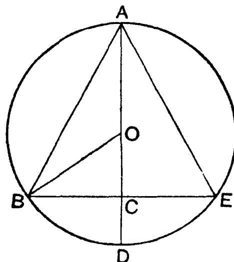


Fig. 10.

Let ABE (Fig. 10) be a section of the cone by a plane through its vertex A and the centre O of the sphere. The volume of the cone is $\frac{1}{3}\pi BC^2 . AC$.

Let $OC=x$; then $AC=R+x$ and $BC^2=R^2-x^2$. The volume is therefore

$$\frac{\pi}{3}(R^2-x^2)(R+x)=\frac{\pi}{3}(R^3+R^2x-Rx^2-x^3),$$

and, since $\pi/3$ is a constant, the volume will be a maximum when the expression in the bracket is a maximum. The problem therefore becomes: if $f(x)=R^3+R^2x-Rx^2-x^3$, find the maximum value of $f(x)$.

We have $\qquad f'(x)=R^2-2Rx-3x^2=(R+x)(R-3x),$

and $f'(x)=0$ when $x=-R$ and when $x=\frac{1}{3}R$.

The value $x=-R$ makes $f(x)$ equal to 0; the value $\frac{1}{3}R$ makes $f(x)$ a maximum. It is, in fact, obvious that there must be a maximum, and it can only be the value given by $x=\frac{1}{3}R$. The maximum cone has the volume $32\pi R^3/81$ or $1{\cdot}24R^3$.

Note that for the maximum cone

$$\sin BAC=1/\sqrt{3} \text{ and } \angle BAC=35^\circ\ 16'.$$

Example 4. Find the maximum value of the curved surface of a cone inscribed in a sphere of radius R.

The curved surface (Fig. 10) is $\pi BC\,.\,AB$ and, in the notation of example 3, is equal to

$$\pi\sqrt{(R^2-x^2)}\sqrt{\{(R^2-x^2)+(R+x)^2\}} \text{ or } \pi\sqrt{(2R)}\sqrt{(R^3+R^2x-Rx^2-x^3)}.$$

Now, since we are only concerned in this case with *positive* quantities, the curved surface will be a maximum when *its square* is a maximum. But the square is $2\pi^2Rf(x)$, where $f(x)$ has the same meaning as in example 3. Hence, the factor $2\pi^2R$ being a constant, the curved surface will be a maximum for the same value of x, namely $\frac{1}{3}R$, as in example 3; the maximum value is $8\pi R^2/3\sqrt{3}$ or $4{\cdot}84R^2$.

EXERCISES. III.

Note. The cylinders and cones referred to in the examples are right circular cylinders and cones.

Find the turning values, stating their nature, and the points of inflexion on the graphs of the functions 1-18.

1. $10+15x-3x^2$. **2.** $(2x+3)(5-x)$.

3. x^3-3x+2. **4.** $4x^3-15x^2+12x-2$.

5. $2x^3-3x^2-12x+10$. **6.** $3x^4-8x^3-6x^2+24x-11$.

7. $3x^4-4x^3-10$. **8.** $x^5-5x^4+5x^3+10$.

9. $(1+x)^2(1-x^2)$. **10.** $x^3(1-x)^2$.

11. $(x-1)(x-2)(x-3)$. **12.** $x(9-x^2)$.

13. $x^2(9-x^2)$. **14.** $x^3(9-x^2)$.

15. $(2x^2+3x+18)/x$. **16.** $(x^2+x+2)/(x-1)$.

17. $(2x^3+3x^2+8)/x^2$. **18.** $(7x^4-30x^3+11x^2-8)/x^2$.

19. Show that the altitude of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $2R/\sqrt{3}$.

20. Show that the curved surface of a cylinder inscribed in a sphere of radius R is a maximum when the altitude of the cylinder is $\sqrt{2} \cdot R$.

21. Show that the altitude of the cylinder of maximum volume that can be inscribed in a cone, whose altitude is h and the radius of whose base is R, is $\frac{1}{3}h$.

22. Show that the curved surface of a cylinder inscribed in the cone of example 21 is a maximum when the altitude of the cylinder is $\frac{1}{2}h$.

23. A cylinder is to be constructed and its *total* surface is to be A square inches; show that the altitude of the cylinder of greatest volume that satisfies the condition is twice the radius of its base.

[Let the altitude of the cylinder be y in. and the radius of its base x in.; then its volume is $\pi x^2 y$ cub. in. and its total surface $2\pi xy + 2\pi x^2$ sq. in. But, by the given condition,

$$2\pi xy + 2\pi x^2 = A,$$

so that $$\pi xy = \tfrac{1}{2}(A - 2\pi x^2), \quad \pi x^2 y = \tfrac{1}{2}(Ax - 2\pi x^3).$$

The problem therefore reduces to that of finding the maximum value of $\frac{1}{2}(Ax - 2\pi x^3)$.]

24. The strength of a rectangular beam varies as the product of the breadth and the square of the depth of the beam; find the breadth and depth of the strongest rectangular beam that can be cut from a cylindrical log, the diameter of whose cross-section is d inches.

25. From a rectangular sheet of tin, the sides of which are a and b inches, equal squares are cut off at each corner and a box with open top is formed by turning up the sides. Find the side of the square so that the box may have maximum content.

26. If $x + y = k$, a constant, find the maximum value of x^3y^2, all the quantities being positive.

Prove from your result that

$$\left(\frac{x}{3}\right)^3\left(\frac{y}{2}\right)^2 < \left(\frac{x+y}{5}\right)^5; \quad a^3b^2 < \left(\frac{3a+2b}{5}\right)^5$$

unless $x/3 = y/2$ or $a = b$, in which case there is equality.

27. The same problem as in example 26 for x^5y^3. Deduce

$$\left(\frac{x}{5}\right)^5\left(\frac{y}{3}\right)^3 < \left(\frac{x+y}{8}\right)^8; \quad a^5b^3 < \left(\frac{5a+3b}{8}\right)^8$$

unless $x/5 = y/3$ or $a = b$, in which case there is equality.

What would you infer the inequalities deduced from x^my^n to be?

CHAPTER III.

DIFFERENTIALS. HIGHER DERIVATIVES.

22. Differentials. In Fig. 11 let $OM = x$, $MP = y = f(x)$, $MN = PR = \delta x$, $RQ = \delta y$ and let the tangent PT cut NQ at T; PR is parallel to MN.

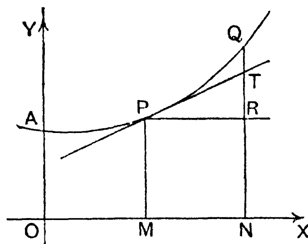


Fig. 11.

The gradient of the curve at P is RT/PR, and its value is the value of dy/dx or $f'(x)$ when $x = OM$.

Now suppose that a point moves along the curve AP, but, when it comes to P, suppose it to proceed, not along the arc PQ but along the tangent PT. The increment of the ordinate of the moving point is, on this supposition, not RQ or δy but RT. This **hypothetical increment** of y is called the **differential** of y and is denoted by dy (or $df(x)$, when $f(x)$ is used in place of y); the symbol dy must, like δy, be taken as a whole.

We thus have

$$\delta y = \text{actual increment of the function } y = RQ,$$
$$dy = \text{differential of the function } y = RT.$$

The hypothetical increment of the *independent* variable x is the same as the actual increment MN or δx; we may therefore write dx, the differential of x, in place of δx, thus securing a symmetrical notation.

Since $$f'(x)=\frac{RT}{PR}=\frac{dy}{dx}=\frac{df(x)}{dx}$$

we deduce $$dy=df(x)=f'(x)dx, \qquad \text{.....................(1)}$$

so that $f'(x)$ is the coefficient of the differential dx; hence the name "differential coefficient."

From (1) we see that **the differential of a function $f(x)$ is equal to the derivative $f'(x)$, multiplied by the differential dx of the independent variable x.**

For example,

$$d(x^2)=2xdx\ ;\ d(x^n)=nx^{n-1}dx\ ;\ d(3x^2-x+4)=(6x-1)dx.$$

Instead of the brackets, a point is often used when the function contains only one term: thus, $d\,.\,x^2$, $d\,.\,x^n$. The notation dx^2, without brackets or point, is usually taken to mean $(dx)^2$, that is, the square of dx.

When the independent variable is, say, t or u we have

$$d\,.\,t^2=2tdt\ ;\ d(\sqrt{t})=\frac{1}{2\sqrt{t}}dt=\frac{dt}{2\sqrt{t}}\ ;\ d(u^3-u)=(3u^2-1)du.$$

23. Approximations. In the definition of dx there is no limitation as to its *magnitude*; we may choose dx of any size we please but, when dx has been chosen, the magnitude of dy becomes definite because $dy=f'(x)dx$. In practice, however, dx (and therefore also dy) is usually supposed to be *small*; when dx is small, dy is a good *approximation* to the value of δy, as will now be proved.

Let the value of $\delta y_1/\delta x_1$ in § 17 be examined; it will be seen that in every case it consists of two terms, namely (i) the derivative, (ii) a term of the form $A\,\delta x_1$. We therefore have, for any value of x, an equation of the form

$$\frac{\delta y}{\delta x}=f'(x)+A\,\delta x.$$

Multiplying by δx we find, since $\delta x=dx$ and $f'(x)dx=dy$,

$$\delta y=f'(x)\delta x+A(\delta x)^2=dy+A(dx)^2.$$

Now, the fractional error involved in replacing δy by dy is $(\delta y-dy)/\delta y$.

But $\frac{\delta y - dy}{\delta y} = \frac{A(dx)^2}{f'(x)dx + A(dx)^2} = \frac{A}{f'(x) + Adx} \times dx.$

If, therefore, dx is small the fractional error will usually be small, because A and $f'(x)$ are usually finite quantities. Of course, in all approximations it is, as a rule, the fractional and not the absolute error that is of importance.

If then $y = f(x)$ and $y + \delta y = f(x + dx)$ we have as approximate equations, when dx is small,

$$f(x+dx) - f(x) = \delta y = dy = f'(x)dx, \quad \ldots\ldots\ldots\ldots(1)$$

$$f(x+dx) = f(x) + f'(x)dx, \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

where, it will be noted, the term in $(dx)^2$ does not appear.

Geometrically, RQ (Fig. 11) is approximately equal to RT when MN is small.

Example. Let $f(x) = x^2$; then $dy = f'(x)dx = 2xdx$.

If $x = 2$ and $dx = 0{\cdot}1$, then $dy = 0{\cdot}4$, $\delta y = 0{\cdot}41$.

If $x = 2$ and $dx = 0{\cdot}01$, then $dy = 0{\cdot}04$, $\delta y = 0{\cdot}0401$.

When $dx = 0{\cdot}1$ the absolute error in taking dy for δy is 0·01, the fractional error is 1/41 and the percentage error about $2\frac{1}{2}$.

When $dx = 0{\cdot}01$, these errors are respectively 0·0001, 1/401 and about $\frac{1}{4}$.

The errors involved in taking $f(x) + f'(x)dx$ in place of $f(x + dx)$ in the two cases are 0·01, 1/441, 0·23 and 0·0001, 1/40401, 0·0025 respectively.

Examples 26–29 of Exercise IV. should be noted.

A very useful case of (2) is obtained by putting $f(x) = x^n$; writing h for dx we find when h is small

$$(x+h)^n = x^n + nx^{n-1}h. \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

The approximate values given by equations (1), (2), (3) may be called *first approximations* to $\delta y, f(x + dx), (x + h)^n$ respectively.

As a method of approximation, the principle of rejecting squares and higher powers of small quantities, such as dx is supposed to be, is of great importance. It may be noted, as a hint that can often be turned to good account, that **in finding the limit of $\delta y_1/\delta x_1$ we may at the outset reject the squares and higher powers of** δx_1, for the simple reason that these have no influence **on the limit.** Thus, in § 17 (3) the terms denoted by $A(\delta x_1)^2$ might have been rejected at the start; because in equation (4) they appear as $A\delta x_1$, and the **limit** of $A\delta x_1$ is zero. Terms such as $A(\delta x_1)^2$ are sometimes

said to be neglected as being "infinitely small" in comparison with those retained; but this phraseology is apt to mislead the beginner, especially when associated with a doctrine of different kinds of "nothing." For a more detailed treatment of this matter, see the author's *Calculus*, §§ 86, 87.

24. Higher Derivatives. The derivative of a function y of x is usually itself a function of x and therefore has a derivative. Thus, if $y=x^4$, then $Dy=4x^3$; but $D(4x^3)=12x^2$, that is, the derivative of the derivative of x^4 is $12x^2$. The function $D(4x^3)$ or $12x^2$ is therefore called the **second derivative** of x^4. Similarly, $D(12x^2)$, that is $24x$, is called the **third derivative** of x^4 and so on. $D(x^4)$ may now, for distinction, be called the first derivative of x^4.

The notation for the higher derivatives is modelled on the notation for indices. Thus,

the first	derivative of y is Dy;		
the second	....	$D(Dy)$,	written D^2y;
the third	....	$D(D^2y)$,	.... D^3y;
the nth	....	$D(D^{n-1}y)$,	.... D^ny.

The suffix $x, t, \ldots$ is inserted when the argument has to be specified; thus, $D_x{}^2y$, $D_x{}^3y \ldots D_t{}^2y$, $D_t{}^3y \ldots$.

In the d/dx notation these derivatives are written thus:

$$D^2y=\frac{d^2}{dx^2}y=\frac{d^2y}{dx^2};\quad D^3y=\frac{d^3}{dx^3}y=\frac{d^3y}{dx^3}$$

and so on.

The accent notation is also used; thus $f''(x)$ means the 2nd derivative of $f(x)$, $f'''(x)$ the 3rd derivative and so on. The nth derivative is $f^{(n)}(x)$, the n being put in brackets.

The student should note examples 2 and 3.

Example 1. If $y=ax^4+bx^3+cx^2+dx+e$ find the first four derivatives of y.

$$Dy \text{ or } \frac{dy}{dx}=4ax^3+3bx^2+2cx+d;$$

$$D^2y \text{ or } \frac{d^2y}{dx^2}=12ax^2+6bx+2c;$$

$$D^3y \text{ or } \frac{d^3y}{dx^3}=24ax+6b;\quad D^4y \text{ or } \frac{d^4y}{dx^4}=24a.$$

Since D^4y is a constant, the 5th and all higher derivatives are in this case zero.

Example 2. If $f'(x)=0$ when $x=a$, show that $f(a)$ will be a maximum value of $f(x)$ provided $f''(a)$ is a negative (not zero) number, but that $f(a)$ will be a minimum value of $f(x)$ provided $f''(a)$ is a positive (not zero) number.

Let h be a small positive number, then, provided $f''(a)$ is not zero, the function $f''(x)$ will have *the same sign* when x is equal to $a-h$ or to $a+h$ as it has when x is equal to a; because the three values of $f''(x)$ cannot differ much from each other.

Now, $f(a)$ is a maximum value of $f(x)$ if, and only if, the gradient $f'(x)$ is positive when $x=a-h$ and negative when $x=a+h$. Hence, as x increases from $a-h$ to $a+h$, the function $f'(x)$ decreases (algebraically), and therefore (§ 19) the derivative of $f'(x)$ must be negative. But the derivative of $f'(x)$ is $f''(x)$, and therefore, as x increases from $a-h$ to $a+h$, $f''(x)$ must be negative; that is, $f''(a)$ must be negative.

In the same way the other part of the theorem is proved; $f'(x)$ is now an increasing function.

If $f''(a)=0$ the reasoning fails; it will be a good exercise to show precisely where it fails.

We have here obtained a **test for a maximum or minimum** that is sometimes useful.

Thus, in § 21, example 1, $f''(x)=24x$. Now,

$$f''(-1{\cdot}5)=-36, \quad f''(1{\cdot}5)=36,$$

so that $f(-1{\cdot}5)$ is a maximum and $f(1{\cdot}5)$ a minimum value of $f(x)$.

Example 3. Show that the abscissae of the points of inflexion on the graph of $f(x)$ are the roots of the equation $f''(x)=0$.

At a point of inflexion the gradient $f'(x)$ has a turning value (§ 19, example 3); therefore the abscissa of such a point must make the derivative of $f'(x)$ equal to zero, that is, it satisfies the equation $f''(x)=0$.

To be certain that a point of inflexion is present, the test that $f'(x)$ has a turning value should be applied.

EXERCISES. IV.

Write down the differentials of the functions 1–13.

1. $2x+3$. **2.** $ax+b$. **3.** $3x^2+1$. **4.** ax^2+b.

5. x^2+4. **6.** x^2+a^2. **7.** a^2-x^2.

8. $3x^2-4x+7$. **9.** ax^2+bx+c. **10.** $\sqrt{x}$.

11. $\frac{1}{\sqrt{x}}$. **12.** $ax^2+\frac{b}{x^2}$. **13.** $ax^n+\frac{b}{x^n}$.

Write down functions of which the functions 14-25 are the differentials.

14. $xdx.$ **15.** $(x+1)dx.$ **16.** $(3x+1)^2dx.$

17. $\frac{dx}{x^2}.$ **18.** $\frac{dx}{\sqrt{x}}.$ **19.** $\frac{dv}{v^{\frac{3}{2}}}.$

20. $\frac{dv}{v^{1\cdot4}}.$ **21.** $\frac{dv}{(v+1)^2}.$ **22.** $\frac{dv}{\sqrt{v+1}}.$

23. $\left(u+2+\frac{1}{u^2}\right)du.$ **24.** $\frac{z^3-2}{z^2}dz.$ **25.** $\frac{y+1}{\sqrt{y}}dy.$

26. If $f(x)=3x^2-x-3$ find the first approximation to $f(x+dx)$.

One root of $3x^2-x-3=0$ is 1·18 approximately; calculate a better approximation. See **Newton's Rule**, page 218.

[Let $1\cdot18+h$ be the root; then $f(1\cdot18+h)=0$. But, since h is small,

$$f(1\cdot18+h)=f(1\cdot18)+f'(1\cdot18)h,$$

so that

$$f(1\cdot18)+f'(1\cdot18)h=0,\ h=-\frac{f(1\cdot18)}{f'(1\cdot18)}.$$

Now, $f(1\cdot18)=-0\cdot0028$, $f'(1\cdot18)=6\cdot08$, so that $h=0\cdot00046$ and the root is 1·18046. Note that in § 22 we may if we please suppose dx to be negative, for Q may be taken on the other side of P.]

27. The equation $3x^3-4x-5=0$ has one real root which is, approximately, equal to 1·5; find a better approximation.

28. The real root of the equation $3x^3+5x-40=0$ is 2·13 approximately; find a better approximation.

29. One root of the equation $x^3-4x^2-7x+24=0$ is 2·18 approximately; find a better approximation.

30. Show that equation (2), § 23, may be written in the forms

$$\text{(i) } f(b)=f(a)+(b-a)f'(a),$$
$$\text{(ii) } f(x)=f(0)+xf'(0).$$

Illustrate by graphs.

Apply § 23 (3), or Example 30, to prove the approximate equations in Examples 31-36; test the approximations in any way you can.

31. $\sqrt{(1+x)}=1+\frac{1}{2}x.$ **32.** $\sqrt{(1+x^2)}=1+\frac{1}{2}x^2.$

33. $\sqrt{(a+x)}=\sqrt{a}+\frac{x}{2\sqrt{a}}.$ **34.** $\sqrt{(p^2+x^2)}=p+\frac{x^2}{2p}.$

35. $\frac{1}{\sqrt{(p^2+x^2)}}=\frac{1}{p}-\frac{x^2}{2p^3}.$

36. $\frac{1}{x+\sqrt{(p^2+x^2)}}=\frac{1}{p}-\frac{x}{p^2}+\frac{x^2}{2p^3}.$

Write down the second and third derivatives of the functions 37–48.

37. $3x^2-5x+7$.

38. $10x^4-15x^3+12x^2+8x-5$.

39. $\frac{1}{x}$.

40. $\frac{1}{x^2}$.

41. $\frac{1}{x+1}$.

42. $\frac{1}{(x+1)^2}$.

43. $ax+b+\frac{c}{x}+\frac{d}{x^2}$.

44. $\sqrt{x}$.

45. $\frac{1}{\sqrt{x}}$.

46. $\frac{1}{x^n}$.

47. $x^2(a-x)^2$.

48. $x^3(a-x)^3$.

Find the points of inflexion on the graphs of the functions 49–53.

49. $2x^3-7x+3$.

50. $4x^3-27x+5$.

51. $5x^3+4x^2-3x+2$.

52. $(x+1)(x+2)(x+3)$.

53. $x^2(a-x)^2$.

54. Show that (with the usual disposition of the axes) the graph of $f(x)$ is **concave upwards** for all values of x that make $f''(x)$ positive, but is **convex upwards** for all values of x that make $f''(x)$ negative. (See § 9.)

CHAPTER IV.

APPLICATIONS TO MECHANICS. FURTHER THEOREMS ON DIFFERENTIATION.

25. Applications to Mechanics. The gradient at a point P on the graph of a function y or $f(x)$ measures *the rate* at which the function increases with respect to x, for the particular value of x which is the abscissa of P (§ 19). The characteristic word for expressing a rate is *per*. Thus, if the gradient of a road is 1/20 the road rises 1/20th of the vertical unit *per unit* of horizontal advance, or 1 vertical unit *per* 20 horizontal units. Again, the speed of a train may be 30 miles *per* hour; the rate at which the velocity of a falling body is accelerated is 32 units of velocity (feet *per* second) *per* second, and so on.

In algebraical language, if, as the independent variable increases from x to $x+\delta x$, the function increases from y to $y+\delta y$ then $\delta y/\delta x$ measures the *average* rate at which y increases for the increment δx; the *limit* of $\delta y/\delta x$ (which is denoted by dy/dx) measures *the* rate at which y increases, for the value x of the independent variable. If y is, for example, a number of square inches and x a number of seconds, then dy/dx means "so many square inches per second"; if x denotes a number of degrees of temperature, then (y being a number of square inches) dy/dx means "so many square inches per degree of temperature," and so on. The student should accustom himself to thinking of this meaning of a derivative.

We shall now take some examples.

Example 1. **Velocity, Acceleration, Force.** At time t (seconds) from a chosen instant, a particle of mass m (pounds), which is moving along a straight line, is at the distance x (feet) from a fixed point O on the line. If the velocity of the particle is v (feet per second), the acceleration a (feet per second per second) and the force acting on it F (poundals), express v and a as t-derivatives, and write down the equation connecting F, m, a.

The velocity at time t is the rate at which the distance x is increasing. When t increases to $t+\delta t$ let x increase to $x+\delta x$; of course, if x decreases the increment δx is negative. The average velocity during the interval δt is $\delta x/\delta t$; and the velocity *at time t, from which the interval δt begins*, is the limit of $\delta x/\delta t$ for δt converging to zero.

Therefore $$v=\frac{dx}{dt}=D_t x. \qquad \text{(i)}$$

(To save needless multiplication of symbols we have not taken a value t_1, as we did in finding derivatives; we suppose for the moment that t has some definite value and we hold that value fixed while finding the limit. The *process* is identical with that adopted before.)

Again, suppose that during the interval δt the velocity takes the increment δv; then $\delta v/\delta t$ is the average rate of increase of v, that is the average acceleration, during that interval. Hence

$$a=\frac{dv}{dt}=\frac{d^2x}{dt^2}. \qquad \text{(ii)}$$

Newton called v the **fluxion** of x, and a the **fluxion** of v or the second fluxion of x; x and v he called the **fluents**. His notation for a fluxion is a dot placed over the fluent; thus,

$$v=\dot{x}\,;\quad a=\dot{v}=\ddot{x}. \qquad \text{(iii)}$$

This notation is still used frequently in mechanics when the **time** is the independent variable.

By the second law of motion, the force F (poundals) in the direction of motion is the time-rate of change of the momentum mv. Therefore

$$F=\frac{d(mv)}{dt}=m\frac{dv}{dt}=ma. \qquad \text{(iv)}$$

Example 2. **Pressure at a point in a fluid.** Let the pressure exerted on either side of a plane area, s square inches, drawn in a fluid be p pounds; the average pressure per square inch over the area is therefore p/s. If P is a point within the area, the pressure of the fluid *at P* is the limit of p/s as the area s tends to zero, the point P being always within the area. The pressure at a point is thus *a rate of pressure*, and is expressed as so many pounds per square inch; instead of "pressure at a point" the phrase "the intensity of pressure at a point" is frequently used.

Example 3. **Elasticity of volume of a fluid** Let p (pounds per square inch) be the intensity of pressure and v (cubic inches) the volume of unit mass of a fluid, p being a definite function of v. When

p increases by δp let v increase by δv; if δp is positive, δv will be negative. The quotient $-\delta v/v$ is called the **mean compression**, and the limit of the quotient of the increment of pressure by the mean compression produced is called **the elasticity of volume of the fluid.** Hence the elasticity of volume is

$$\underset{\delta v=0}{L}\ \delta p \div \left(-\frac{\delta v}{v}\right) = \underset{\delta v=0}{L} - v\frac{\delta p}{\delta v} = -v\frac{dp}{dv}.$$

When (i) $pv=C$, (ii) $pv^\gamma=C$ we obtain (§ 18, Example 3) (i) p, (ii) γp for the elasticity of volume.

Example 4. If W is the work done in stretching a rod or string from its natural length a to the length $a+x$, find dW/dx, assuming Hooke's law to hold.

The quotient x/a is called the *unital extension*, and by Hooke's law the force F required to produce that extension is proportional to it, so that $F=Ex/a$ where E is a constant. When x has become $x+\delta x$ let the force be $F+\delta F$ and the work done $W+\delta W$. The work done, δW, in increasing x by δx will lie between $F\delta x$ and $(F+\delta F)\delta x$; hence $\delta W/\delta x$ lies between F and $F+\delta F$. But when δx converges to zero so does δF, and therefore

$$\frac{dW}{dx} = F = E\frac{x}{a}.$$

It is easy to verify that $W=\frac{1}{2}Ex^2/a=\frac{1}{2}xF$; the work will be expressed in inch-pounds if x is given in inches and F in pounds.

It will be noticed that $(F+\delta F)\delta x$ differs from $F\delta x$ by the product $\delta F.\delta x$; but, when we divide by δx and then take the limit, the term δF will disappear. In this and similar cases, then, we might reject the product $\delta F.\delta x$ at the start and take δW equal to $F\delta x$, the *differential* of W. In practice this rejection is usually made to begin with; this procedure simplifies the work and *neglects nothing* though it *rejects something* !

26. Additional Theorems of Differentiation. We shall now consider some cases of differentiation to which the results given in § 18 are not immediately applicable.

I. Function of a Function. Suppose $y=(x^2-x+1)^{\frac{1}{2}}$. The base (x^2-x+1) of the power is neither x simply nor a linear function of x, and therefore the derivative of y can not be obtained either by rule I. or by rule V. of § 18. In a case like this we proceed as follows: put a single variable, u say, for the base (x^2-x+1); y then becomes a function of u but, through u, is a function of x. The *two* equations

$$y=u^{\frac{1}{2}},\ u=x^2-x+1$$

express y as *a function of a function* of x.

Now, we can calculate dy/du and du/dx; how are these two derivatives connected with dy/dx? The answer is that dy/dx is equal to their product, as may be proved thus:

When x takes the increment δx let u take the increment δu, and when u takes the increment δu let y take the increment δy; to the increment δx of x, therefore, corresponds the increment δy of y. But, by ordinary algebra,

$$\frac{\delta y}{\delta x}=\frac{\delta y}{\delta u}\times\frac{\delta u}{\delta x},$$

and therefore, taking the limit of each member of the equation for δx (and therefore also δu) converging to zero, we deduce

$$\mathbf{\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}}. \qquad \text{(A)}$$

The theorem (A) is clearly quite general. To calculate dy/du we must of course express y in terms of u.

Example 1. Find dy/dx when $y=(x^2-x+1)^{\frac{1}{2}}$.

As above, $y=u^{\frac{1}{2}},\ u=x^2-x+1$;

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=\tfrac{1}{2}u^{-\frac{1}{2}}\times(2x-1)=\frac{2x-1}{2(x^2-x+1)^{\frac{1}{2}}}.$$

With a little practice the substitution of u can be done *mentally*; thus,

$$D(x^2-a^2)^{\frac{1}{2}}=\tfrac{1}{2}(x^2-a^2)^{-\frac{1}{2}}\times 2x=x/(x^2-a^2)^{\frac{1}{2}}.$$

Example 2. An important application of (A) occurs in mechanics. Using the notation of § 25, example 1, we have

$$a=\frac{dv}{dt}=\frac{dv}{dx}\times\frac{dx}{dt}=\frac{dv}{dx}\times v=v\frac{dv}{dx}.$$

But,

$$\frac{d\,.\,v^2}{dx}=\frac{d\,.\,v^2}{dv}\times\frac{dv}{dx}=2v\frac{dv}{dx},$$

so that

$$a=\tfrac{1}{2}\frac{d\,.\,v^2}{dx};\quad F=ma=\tfrac{1}{2}m\frac{d\,.\,v^2}{dx}=\frac{d(\tfrac{1}{2}mv^2)}{dx}.$$

Equation (iv) of § 25, example 1, may therefore be put in the two forms

$$F=\frac{d(mv)}{dt}\ \text{.........(iv)};\quad F=\frac{d(\tfrac{1}{2}mv^2)}{dx}.\ \text{...............(iv}a\text{)}$$

In (iv) F is expressed as the **time-rate** of change of **momentum**; in (iv$a$) as **the space-rate** (rate per unit of distance) of change of **kinetic energy**. (The kinetic energy is $\tfrac{1}{2}mv^2$.)

In the same way as theorem (A) is established we prove

$$\frac{dy}{dx} \times \frac{dx}{dy} = 1 \quad \text{or} \quad \frac{dx}{dy} = 1 \div \frac{dy}{dx}. \quad \ldots\ldots\ldots\ldots\ldots\ldots (\text{A}')$$

Theorem (A′) has a simple graphical interpretation which the student should find. (Take dy/dx as the gradient of the graph of y and use the *trigonometrical* interpretation of the gradient, § 5.)

II. **Derivative of a Product.** Let u and v be functions of x. When x takes the increment δx, let u take the increment δu and v the increment δv. Then the increment $\delta(uv)$ of the product uv is

$$\delta(uv) = (u + \delta u)(v + \delta v) - uv = v\delta u + u\delta v + \delta u \times \delta v.$$

Therefore, $$\frac{\delta(uv)}{\delta x} = v\frac{\delta u}{\delta x} + u\frac{\delta v}{\delta x} + \frac{\delta u}{\delta x} \times \delta v,$$

so that, $$\frac{d(uv)}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\text{B})$$

because the limit of $\frac{\delta u}{\delta x} \times \delta v$ is $\frac{du}{dx} \times 0$, which is zero.

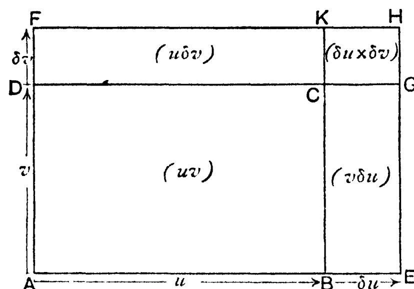


Fig. 12.

Fig. 12 gives a geometrical interpretation. The differential vdu is equal to the rectangle BG, the differential udv to the rectangle DK, and the differential $d(uv)$ is equal to the sum of BG and DK.

The increment $\delta(uv)$ of the rectangle AC (or uv) differs from the differential $d(uv)$ by the rectangle CH which is equal to $\delta u \times \delta v$.

If u, v are the lengths, in inches say, of the sides of a rectangular plate $ABCD$ when the temperature of the plate is x degrees, then $d(uv)/dx$ is the rate, in square inches per degree, at which the area is

expanding; du/dx and dv/dx are the rates, in inches per degree, at which the sides are expanding.

If there are more than two factors, say u, v, w, we may extend theorem (B) by applying it twice; thus, first consider vw as forming one factor and we find

$$D(uvw) = D(u\,.\,vw) = vwDu + uD(vw).$$

But $$D(vw) = wDv + vDw,$$

so that $$D(uvw) = vwDu + uwDv + uvDw. \quad\text{............(B}'\text{)}$$

If we divide both members of (B′) by uvw we obtain

$$\frac{D(uvw)}{uvw} = \frac{Du}{u} + \frac{Dv}{v} + \frac{Dw}{w}, \quad\text{..................(B}''\text{)}$$

a form which is easily extended to any number of factors.

Example 3. Differentiate $(2x+3)\sqrt{(x^2-x+1)}$.

$$\begin{aligned} D(2x+3)\sqrt{(x^2-x+1)} &= \sqrt{(x^2-x+1)}\,D(2x+3) + (2x+3)\,D\sqrt{(x^2-x+1)} \\ &= \sqrt{(x^2-x+1)} \times 2 + (2x+3) \times (2x-1)/2\sqrt{(x^2-x+1)} \\ &= (8x^2+1)/2\sqrt{(x^2-x+1)}, \end{aligned}$$

the value of $D\sqrt{(x^2-x+1)}$ being taken from example 1. To lessen the chance of mistakes in complicated differentiations, a derivative that cannot be at once written down should be worked out separately and then inserted.

III. **Derivative of a Quotient.** Let the quotient be u/v and write the quotient *as a product*, namely uv^{-1}; then by (B)

$$\frac{d(u/v)}{dx} = \frac{d(uv^{-1})}{dx} = v^{-1}\frac{du}{dx} + u\frac{d(v^{-1})}{dx}.$$

But $$\frac{d(v^{-1})}{dx} = \frac{d(v^{-1})}{dv} \times \frac{dv}{dx} = -v^{-2}\frac{dv}{dx} = -\frac{1}{v^2}\frac{dv}{dx}.$$

Substituting and reducing, we find

$$\mathbf{D}\left(\frac{\mathbf{u}}{\mathbf{v}}\right) = \frac{\mathbf{vDu - uDv}}{\mathbf{v}^2}. \quad\text{.....................(C)}$$

Example 4. Differentiate $(7x^2-3x+1)/(x^2+2x+4)$.

$$\begin{aligned} D\left(\frac{7x^2-3x+1}{x^2+2x+4}\right) &= \frac{(x^2+2x+4)(14x-3)-(7x^2-3x+1)(2x+2)}{(x^2+2x+4)^2} \\ &= \frac{17x^2+54x-14}{(x^2+2x+4)^2}. \end{aligned}$$

When the denominator is a power it is better to express the quotient as a product and to use (B) and (A).

The following example is of a type that is of great use in integration. The process is that of **changing the independent variable.**

Example 5. Given $\frac{dy}{dx}=x\sqrt{(x^2+1)}$ find $\frac{dy}{du}$ where $u=x^2+1$, expressing the value in terms of u. Deduce the integral of $x\sqrt{(x^2+1)}$.

By (A), (A′), $\frac{dy}{du}=\frac{dy}{dx}\times\frac{dx}{du}=\frac{dy}{dx}\div\frac{du}{dx}$

because $\frac{dx}{du}=1\div\frac{du}{dx}$.

Hence, $\frac{dy}{du}=x\sqrt{(x^2+1)}\div 2x=\frac{1}{2}\sqrt{(x^2+1)}=\frac{1}{2}\sqrt{u}$.

The integral with respect to u of $\frac{1}{2}\sqrt{u}$ is $\frac{1}{3}u^{\frac{3}{2}}$, as may be tested by differentiating with respect to u. Therefore $y=\frac{1}{3}u^{\frac{3}{2}}$, so that, replacing u by x^2+1 we find $y=\frac{1}{3}(x^2+1)^{\frac{3}{2}}$. The student should test that the x-derivative of $\frac{1}{3}(x^2+1)^{\frac{3}{2}}$ is $x\sqrt{(x^2+1)}$.

The next example shows how to find the gradient of a curve when the coordinates are expressed in terms of a third variable as is so often the case in mechanics.

Example 6. If x and y are both functions of a third variable t, show that
$$\frac{dy}{dx}=\frac{dy}{dt}\div\frac{dx}{dt}.$$

We can apply (A) and (A′); for y is a function of t, and t is a function of x, since x is a function of t. Therefore
$$\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}=\frac{dy}{dt}\div\frac{dx}{dt}.$$

Or, we may use differentials. If t takes the increment dt then the *differentials* dy, dx are given by
$$dy=\frac{dy}{dt}dt,\quad dx=\frac{dx}{dt}dt.$$

Dividing dy by dx, we obtain the result stated.

EXERCISES. V.

1. The horizontal and vertical coordinates (x, y) of a moving point are, at time t,
$$x=400t,\quad y=300t-16t^2.$$

Find the horizontal and vertical components (i) of the velocity, (ii) of the acceleration of the point. Find also the highest point of the path and give the direction of motion of the point (that is, give the gradient of the path) at time t. In what direction is the point moving when $t=0$?

2. The same problem as in example 1 when

$$x = Ut, \quad y = Vt - \tfrac{1}{2}gt^2.$$

3. A point is moving in a circle of radius a, and at time t the radius to the point makes an angle θ with a fixed radius. If v is the speed and ω the angular velocity of the point, show that

$$\omega = \frac{d\theta}{dt}, \quad v = \omega a.$$

Express in symbols the tangential acceleration of the point.

4. If N is the number of lines of force passing through a circuit, state in words the meaning of $-dN/dt$, t denoting the time at which the number of lines is N.

5. Express in symbols the statement that the electromotive force E is the sum of two terms of which the first is the product of the resistance R and the current C, and the second is the product of the self-inductance L and the time-rate of increase of C.

Differentiate the functions 6–47.

6. $\sqrt{(x^2-1)}$.
7. $\sqrt{(1-x^2)}$.
8. $\sqrt{(x^2+a^2)}$.
9. $\sqrt{(a^2-x^2)}$.
10. $\sqrt{(3x^2+5)}$.
11. $\sqrt{(5-3x^2)}$.
12. $\sqrt{(ax^2+b)}$.
13. $\sqrt{(b-ax^2)}$.
14. $\sqrt{(x^2+2x-3)}$.
15. $\sqrt{(3x^2-4x+5)}$.
16. $\sqrt{(5+4x-3x^2)}$.
17. $\sqrt{(ax^2+bx+c)}$.
18. $\dfrac{1}{\sqrt{(x^2+1)}}$.
19. $\dfrac{1}{\sqrt{(1-x^2)}}$.
20. $\dfrac{1}{\sqrt{(5x^2-10x+6)}}$.
21. $\dfrac{1}{\sqrt{(ax^2+bx+c)}}$.
22. $\sqrt[3]{(x^2+1)}$.
23. $\sqrt[3]{(ax^3+b)}$.
24. $\dfrac{1}{\sqrt[3]{(ax^3+b)}}$.
25. $\dfrac{1}{\sqrt[3]{(ax^3+bx+c)}}$.
26. $(ax+b)(Ax^2+Bx+C)$.
27. $(x^2+4x+3)(x^2-4x+3)$.
28. $(x+1)(x+2)(x+3)$.
29. $(2x-1)^2(3x+4)^2$.
30. $(ax+b)^2(cx+d)^3$.
31. $(ax+b)(Ax^2+Bx+C)^2$.
32. $x\sqrt{(1-x^2)}$.
33. $(2x+3)\sqrt{(x^2+4)}$.
34. $(a+x)\sqrt{(a^2-x^2)}$.
35. $(ax+b)\sqrt{(Ax^2+B)}$.
36. $(3x+4)^2\sqrt{(x^2+3x-2)}$.
37. $\sqrt{\{(x+3)(x-2)\}}$.
38. $\dfrac{3x+4}{5x+6}$.
39. $\dfrac{ax+b}{cx+d}$.
40. $\dfrac{x+1}{x^2+1}$.
41. $\dfrac{x^2}{x^2+1}$.
42. $\dfrac{1+3x-x^2}{2+5x+x^2}$.
43. $\dfrac{(3x+2)^2}{2x-x^2}$.
44. $\dfrac{x}{\sqrt{(x+1)}}$.
45. $\sqrt{\left(\dfrac{x-1}{x+1}\right)}$.
46. $\sqrt{\left(\dfrac{x^2+x+1}{x^2-x+1}\right)}$.
47. $\dfrac{x}{\sqrt{(ax^2+bx+c)}}$.

48. At time t seconds the three conterminous edges of a rectangular parallelepiped are u, v, w feet respectively; find the rate at which the volume is increasing. Of which of the formulae in § 26 does this example furnish an illustration?

49. The radius of a circle is increasing at the rate of 2 feet per minute; at what rate is (i) the circumference, (ii) the area, increasing when the radius is 10 feet?

50. If the radius of a circle is increasing at the rate of v feet per minute, at what rate is the area increasing when the radius is x feet?

51. The radius of a sphere is growing at the rate of v feet per minute; at what rate is (i) the surface, (ii) the volume, growing when the radius is x feet?

52. A vessel in the shape of a right circular cone, with vertex downwards and axis vertical, is being filled with water; the inflow of water is at the uniform rate of 12 cubic feet per minute. At what rate is the surface of the water in the vessel rising when its depth is (i) 10 feet, (ii) x feet, the height of the vessel being 20 feet and the radius of its circular top 10 feet?

53. A reservoir has plane sloping sides and ends; its top and bottom are horizontal rectangles of sides 24 and 16 feet, and 12 and 8 feet respectively, and its depth is 40 feet. If water flows into it at the uniform rate of 30 cubic feet per minute, at what rate is the surface rising when the depth of water is 10 feet?

Maxima and Minima.

Find the maximum and minimum values of the functions 54–61; a is a positive constant.

54. $x\sqrt{(25-x^2)}$. **55.** $(a+x)\sqrt{(a^2-x^2)}$. **56.** $x^4(x+1)^3$.

57. $\dfrac{x}{1+x^2}$. **58.** $\dfrac{x}{\sqrt{(1+x^2)}}$. **59.** $\dfrac{x^2+x+1}{x^2-x+1}$.

60. $\dfrac{x}{ax^2+2bx+a}$. **61.** $(a+x)\sqrt{(a-x)}$.

62. The stiffness of a rectangular beam varies as the product of the breadth and the cube of the depth; find the breadth and depth of the stiffest rectangular beam that can be cut from a cylindrical log, the diameter of the cross-section being d inches.

63. Given the *total* surface of a cone, show that when the volume is a maximum the sine of half the vertical angle will be $\frac{1}{3}$.

64. Assuming that the brightness of a small surface A varies inversely as the square of the distance r from the source of light and directly as the cosine of the angle between r and the normal to the surface at A, find at what height above the centre of a horizontal circle of radius a an electric light should be placed so that the brightness at the circumference may be greatest.

65. If the intensities of two sources A, B of light be a^3, b^3 respectively, find the point on the line AB at which the brightness is least.

Change of the Independent Variable.

In examples 66–75, find $\frac{dy}{du}$, expressing its value in terms of u; deduce the value of y in terms of x. Show that example 75 includes the others as particular cases.

66. $\frac{dy}{dx}=(x+1)\sqrt{(x^2+2x+2)}$; $u=x^2+2x+2$.

67. $\frac{dy}{dx}=(2x+3)(x^2+3x-2)^3$; $u=x^2+3x-2$.

68. $\frac{dy}{dx}=x\sqrt{(3-x^2)}$; $u=3-x^2$.

69. $\frac{dy}{dx}=x\sqrt{(ax^2+b)}$; $u=ax^2+b$.

70. $\frac{dy}{dx}=\frac{x}{\sqrt{(x^2+1)}}$; $u=x^2+1$.

71. $\frac{dy}{dx}=\frac{x-1}{\sqrt{(2x^2-4x+1)}}$; $u=2x^2-4x+1$.

72. $\frac{dy}{dx}=\frac{2ax+b}{\sqrt{(ax^2+bx+c)}}$; $u=ax^2+bx+c$.

73. $\frac{dy}{dx}=x^2\sqrt{(ax^3+b)}$; $u=ax^3+b$.

74. $\frac{dy}{dx}=(2ax+b)(ax^2+bx+c)^n$; $u=ax^2+bx+c$.

75. $\frac{dy}{dx}=[f(x)]^n\times f'(x)$; $u=f(x)$.

CHAPTER V.

INTEGRATION OF POWERS.

27. Integration. Suppose we know the gradient at *every* point of a road with no turns in it, and also the height of *one* point of the road above a datum line: we can then, it is clear, draw a contour or trace of the road, and show the height of any point on it above the datum line. Stated geometrically the problem is simply: given the gradient at every point of a plane curve and also the coordinates of one point on it, find the equation of the curve.

Consider the case in which the gradient is given by the equation

$$Dy = 3 + 4x - x^2 \quad \text{.............................(1)}$$

and the curve goes through the point (2, 9).

The first question is, Can we find a function which will, when differentiated, give $3+4x-x^2$? From what we know of differentiation we can say that $3x+2x^2-\frac{1}{3}x^3$ is such a function (test by differentiating it); but, bearing in mind that a constant *term* of a function does not appear in its derivative, we see that, if a constant term be added to the function just found, the function so obtained will also have $3+4x-x^2$ as its derivative. Hence we put

$$y = 3x + 2x^2 - \tfrac{1}{3}x^3 + C \quad \text{.............................(2)}$$

where C is any constant.

For different values of C the equation (2) will give different values of y when $x=2$. To solve the problem, we must choose C so that $y=9$ when $x=2$; that is, equation (2) must be true when $x=2$ and $y=9$. Hence

$$9 = 6 + 8 - \tfrac{8}{3} + C; \quad C = -\tfrac{7}{3}$$

and the equation

$$y = 3x + 2x^2 - \tfrac{1}{3}x^3 - \tfrac{7}{3} \quad \text{.............................(3)}$$

solves the problem completely. The only tests needed are (i) that it gives for Dy the value in (1), and (ii) that it is true when $x=2$ and $y=9$.

The process by which the problem has been solved reduces, mere algebra apart, to finding a function which has a given function as its derivative, and is called **Integration**. Any function whose derivative is equal to a given function is called an **integral** of the given function. When *any constant term*, *C* say, is added to an integral of a function the resulting function is also an integral; it is called the **general integral**, and C is called the **constant of integration**. Thus $\frac{1}{3}x^3$, $\frac{1}{3}x^3+1$, $\frac{1}{3}x^3-2$ are integrals of x^2; $\frac{1}{3}x^3+C$ is the general integral of x^2.

The variable part of an integral is usually called the **indefinite integral**, or simply "the integral." Thus $\frac{1}{3}x^3$ is the integral of x^2; $\frac{1}{3}x^3+1$, $\frac{1}{3}x^3-2$ are particular cases of the general integral obtained by giving C the values 1, -2 respectively.

The notation for the (indefinite) integral of a function

$F(x)$ is $$\int F(x)\,dx;$$

read, "the integral of $F(x)dx$." The differential dx indicates the **variable of integration**, namely x, and the joint symbol $\int \ldots dx$ means "integral of ... with respect to x." The function to be integrated, $F(x)$, is called **the integrand**.

We thus have, for instance:

$$\text{(i)} \int(3+4x-x^2)dx=3x+2x^2-\tfrac{1}{3}x^3; \quad \text{(ii)} \int(1+t)dt=t+\tfrac{1}{2}t^2.$$

The test of the correctness of integration is simply

Derivative of Integral = Integrand,

or, in symbols, $$\frac{d}{dx}\left[\int F(x)dx\right]=F(x).$$

The student will find that his progress will be more rapid if he actually tests his results by differentiating them than if he contents himself with looking up the Answers. Thus the integral in equation (ii) is correct because

$$\frac{d}{dt}(t+\tfrac{1}{2}t^2)=1+t=\text{integrand}.$$

28. Integration of Powers. If n is not equal to -1 we have (§ 18 I.),

$$\int \mathbf{x^n dx} = \frac{\mathbf{x^{n+1}}}{\mathbf{n+1}}. \quad \text{(1)}$$

In words: to integrate x^n, when n is not -1, **add** 1 to the index n and then **divide** by the index so increased.

More generally, when n is not -1, we have (§ 18, V.)

$$\int \mathbf{(ax+b)^n dx} = \frac{\mathbf{(ax+b)^{n+1}}}{\mathbf{(n+1)a}}. \quad \text{(1}a\text{)}$$

The form (1a) occurs so often that it should be committed to memory. The $ax+b$ is a linear function of x and the integral contains a, the coefficient of x, in the denominator.

The above rule fails when $n=-1$; we state here, so that the results may be used in working problems, that

$$\int \frac{\mathbf{1}}{\mathbf{x}}\mathbf{dx} = \mathbf{\log_e x}\,; \quad \int \frac{\mathbf{1}}{\mathbf{ax+b}}\mathbf{dx} = \frac{\mathbf{1}}{\mathbf{a}}\mathbf{\log_e(ax+b)}. \quad \text{(2)}$$

$1/x$ is, of course, the same as x^{-1}. These results are proved in § 61. $\log_e x$ is the *Napierian* logarithm of x (§ 61).

When the integrand is a fraction the symbol dx is often written in the numerator; thus

$$\int \frac{dx}{x^2} = -\frac{1}{x}; \quad \int \frac{dx}{x} = \log_e x.$$

It is evident from the rules of differentiation, that the integral of the sum of two or more terms is equal to the sum of the integrals of the terms; thus,

$$\int (3+4x-x^2)dx = \int 3dx + \int 4xdx - \int x^2 dx$$
$$= 3x + 2x^2 - \tfrac{1}{3}x^3.$$

Also a constant *factor* may be written outside the integral sign; thus

$$\int 4x^2 dx = 4\int x^2 dx = \frac{4}{3}x^3; \quad \int ax^n dx = a\int x^n dx.$$

The constant of integration is not usually inserted when finding indefinite integrals, but it must always be **added** when a problem is being discussed.

Example 1. The acceleration of a particle of mass m which is moving in a straight line is constant, equal to g; if, at time $t=0$, its velocity is V and its distance from a fixed point O on the line a, find its position at time t.

Let x be its distance from O at time t; then by § 25, example 1 (ii),

$$\frac{d^2x}{dt^2}=g. \qquad (1)$$

This equation is called "the differential equation of motion." Now integrate with respect to t; therefore

$$v=\frac{dx}{dt}=gt+C.$$

When $t=0$, $v=V$, so that the constant $C=V$, and we get

$$v=\frac{dx}{dt}=gt+V. \qquad (2)$$

Integrate again; therefore

$$x=\tfrac{1}{2}gt^2+Vt+C'$$

where C' is a constant. But when $t=0$, $x=a$ and therefore $C'=a$. The distance of the point from O at time t is therefore

$$x=\tfrac{1}{2}gt^2+Vt+a. \qquad (3)$$

The kinetic energy E is equal to $\frac{1}{2}mv^2$; we have, combining (2) and (3),

$$E=\tfrac{1}{2}mv^2=\tfrac{1}{2}mV^2+mg(x-a). \qquad (4)$$

Equation (4) may be obtained by equation (iv a), p. 51; for

$$\frac{dE}{dx}=F=mg\,;\quad E=mgx+\text{constant}\,;$$

when $t=0$, $x=a$, $E=\frac{1}{2}mV^2$, so that the constant is $\frac{1}{2}mV^2-mga$, giving equation (4).

Example 2. Integrate $(8x^3+3)/(2x+1)$.

By division we find

$$\frac{8x^3+3}{2x+1}=4x^2-2x+1+\frac{2}{2x+1}\,;$$

therefore, $$\int\frac{8x^3+3}{2x+1}dx=\tfrac{4}{3}x^3-x^2+x+\log(2x+1)$$

where $\log(2x+1)$ means the Napierian logarithm of $2x+1$.

Note. Unless the contrary is stated the symbol "log" is now to be understood as meaning the Napierian logarithm.

Example 3. Integrate $(5x-7)/(x+1)(2x-1)$.

Express the fraction as the sum of two simple fractions, that is, resolve the fraction into its partial fractions. Since the fraction is a

proper fraction (that is, a fraction whose numerator is of a lower degree than its denominator) we write

$$\frac{5x-7}{(x+1)(2x-1)}=\frac{A}{x+1}+\frac{B}{2x-1}, \quad \text{.....................} \text{(i)}$$

and find the values of A and B that make this equation an identity. The simplest method is the following. Clear of fractions; we find

$$5x-7=A(2x-1)+B(x+1). \quad \text{.......................} \text{(ii)}$$

Equation (ii) is an identity, and is therefore true for every value of x First, choose x so that $x+1=0$, that is, $x=-1$; we thus obtain

$$-5-7=A(-2-1) \text{ or } A=4.$$

Next, choose x so that $2x-1=0$, that is, $x=\frac{1}{2}$; therefore

$$\tfrac{5}{2}-7=B(\tfrac{1}{2}+1) \text{ or } B=-3.$$

Hence
$$\frac{5x-7}{(x+1)(2x-1)}=\frac{4}{x+1}-\frac{3}{2x-1},$$

and
$$\int\frac{5x-7}{(x+1)(2x-1)}\,dx=4\log(x+1)-\tfrac{3}{2}\log(2x-1).$$

When the fraction is *improper* (that is, when the degree of the numerator is higher than that of the denominator) we first express the fraction as the sum of an integral function and a proper fraction, and then resolve the proper fraction into its partial fractions.

For example,

$$\frac{2x^3-3x^2-26x+58}{x^2+x-12}=2x-5+\frac{3x-2}{x^2+x-12};$$

and
$$\frac{3x-2}{x^2+x-12}=\frac{1}{x-3}+\frac{2}{x+4},$$

so that the given improper fraction is equal to

$$2x-5+\frac{1}{x-3}+\frac{2}{x+4}$$

of which the integral is

$$x^2-5x+\log(x-3)+2\log(x+4).$$

Example 4. Show that

$$\int\frac{dx}{x^2-a^2}=\frac{1}{2a}\log\left(\frac{x-a}{x+a}\right).$$

We have
$$\frac{1}{x^2-a^2}=\frac{1}{2a}\left(\frac{1}{x-a}-\frac{1}{x+a}\right),$$

and therefore the integral is

$$\frac{1}{2a}\{\log(x-a)-\log(x+a)\} \text{ or } \frac{1}{2a}\log\left(\frac{x-a}{x+a}\right).$$

Integration, like all inverse operations, is essentially a *tentative* process; it is only a comparatively small number of functions whose integrals can be expressed in terms of ordinary functions, and much practice is needed for the acquisition of facility. In this book, however, we deal only with simple cases, and we shall indicate as they occur the more important algebraic and trigonometric theorems that are useful in handling integrals. Here we note the methods of examples 2, 3, 4. The division in example 2 and the resolution into partial fractions in examples 3 and 4 reduce the functions to forms that are *immediately integrable*, that is, that can be integrated by applying a standard result. The standard results obtained up to this stage are given in § 28 (1), (1*a*) and (2); as we proceed we shall add to these *standard forms*. For a full discussion of partial fractions the student must refer to a text-book of algebra.

The following examples are very easy but the student will do well to try most of them, simply to gain *mechanical dexterity*; when he comes to practical work he must not be hampered by imperfect knowledge of the use of his tools.

EXERCISES. VI.

Integrate the functions 1–50:

1. 1. **2.** x. **3.** $x+1$.

4. $2x+1$. **5.** $\frac{1}{2}x+\frac{1}{2}$. **6.** $ax+b$.

7. $5x^2-3x+4$. **8.** $(x-1)(x-2)$. **9.** $(2x-1)(x-2)$.

10. ax^2+bx+c. **11.** $(ax+b)(Ax+B)$. **12.** $x-x^3$.

13. $3+5x+x^2-3x^3$. **14.** ax^3+bx^2+cx+d. **15.** $\sqrt{x}$.

16. $\frac{1}{\sqrt{x}}$. **17.** $\frac{1}{x^2}$. **18.** $\frac{1}{x^{1\cdot41}}$.

19. $\frac{1}{x+1}$. **20.** $\frac{1}{(x+1)^2}$. **21.** $\frac{x+2}{x}$.

22. $\frac{x+2}{x^2}$. **23.** $\frac{3x^2-2x+1}{x^2}$. **24.** $\frac{ax^2+bx+c}{x^2}$.

25. $(2x+1)^3$. **26.** $\frac{1}{(2x+1)^3}$. **27.** $\sqrt{(2x+3)}$.

28. $\frac{1}{\sqrt{(2x+3)}}$. **29.** $\sqrt{(ax+b)}$. **30.** $\frac{1}{\sqrt{(ax+b)}}$.

31. $\dfrac{1}{2x-3}$. 32. $\dfrac{1}{3-2x}$. 33. $\dfrac{x+1}{x-1}$.

34. $\dfrac{2x}{2x-3}$. 35. $\dfrac{x^3+3x^2-4x+5}{x-1}$. 36. $\dfrac{2x^3-3x^2+2}{x+2}$.

37. $\dfrac{1}{x^2-1}$. 38. $\dfrac{1}{x^2-4}$. 39. $\dfrac{1}{x^2-3}$.

40. $\dfrac{1}{4x^2-9}$. 41. $\dfrac{1}{3x^2-7}\left[=\dfrac{1}{3(x^2-7/3)}\right]$. 42. $\dfrac{x}{x^2-4}$.

43. $\dfrac{x}{x^2-a^2}$. 44. $\dfrac{x+1}{(x-2)(x-3)}$. 45. $\dfrac{3x+2}{(x-1)(2x+3)}$.

46. $\dfrac{1}{(x-a)(x-b)}$. 47. $\dfrac{x}{(x-a)(x-b)}$. 48. $\dfrac{3x-7}{(x-1)(x-2)(x-3)}$.

49. $\dfrac{1}{x^2(x-1)}$. 50. $\dfrac{5-2x}{(x-1)^2(x+2)}$.

Solve the differential equations 51-55, determining the constants of integration so as to satisfy the condition stated in each case.

51. $Dy=6-3x$; $y=0$ when $x=0$.

52. $Dy=ax+b$; $y=c$ when $x=0$.

53. $Dy=x-x^2$; $y=1$ when $x=1$.

54. $Dy=x-1/x^2$; $y=2$ when $x=1$.

55. $D^2y=x+1$; $y=0$, $Dy=1$ when $x=0$.

56. At time t the component velocities of a point parallel to the coordinate axes are given by the equations

$$\frac{dx}{dt}=V\cos\alpha,\quad \frac{dy}{dt}=V\sin\alpha-gt\ ;$$

if the point is at the origin when $t=0$, find the values of x and y at time t.

57. At time t the component accelerations of a point parallel to the coordinate axes are given by the equations

$$\frac{d^2x}{dt^2}=0,\quad \frac{d^2y}{dt^2}=-g\ ;$$

find the coordinates of the point at time t given that, when $t=0$,

$$x=0,\quad y=0,\quad \frac{dx}{dt}=V\cos\alpha,\quad \frac{dy}{dt}=V\sin\alpha.$$

58. A particle of mass m is moving in a straight line; when its distance from a fixed point O on the line is x, the force acting on it is $-kmx$, where k is a positive constant. If the kinetic energy E is zero when $x=a$ find the value of E in any other position of the particle.

59. If in example 58 the force is $-km/x^2$ find E, given that E is zero when x is infinite.

The equations 60-63 occur in the theory of the bending of beams, y measuring the deflection of the beam. Integrate the equations, subject to the conditions stated. Find in each case the maximum value of y.

60. $EI\frac{d^2y}{dx^2}=\frac{1}{2}W(\frac{1}{2}L-x)$; $y=0$, $\frac{dy}{dx}=0$ when $x=0$.

61. $EI\frac{d^2y}{dx^2}=-\frac{1}{2}Wx$; $y=0$ when $x=0$, $\frac{dy}{dx}=0$ when $x=\frac{1}{2}L$.

62. $EI\frac{d^2y}{dx^2}=-\frac{1}{2}w(Lx-x^2)$; $y=0$ when $x=0$ and when $x=L$.

63. $EI\frac{d^2y}{dx^2}=K-\frac{1}{2}w(Lx-x^2)$; $y=0$, $\frac{dy}{dx}=0$ when $x=0$, and K is a constant to be determined by the condition that $y=0$ when $x=L$.

29. Change of Variable. When a function is not immediately integrable, it can frequently be reduced to a form that is immediately integrable by a change of the variable of integration. This aid to integration is of the utmost importance.

The method of changing the variable has been illustrated in example 5, § 26, p. 54.

Suppose we have to integrate $x\sqrt{(x^2+1)}$; let

$$y=\int x\sqrt{(x^2+1)}dx, \text{ so that } \frac{dy}{dx}=x\sqrt{(x^2+1)}.$$

Now let $u=x^2+1$; then, by § 26, example 5,

$$\frac{dy}{du}=\frac{1}{2}\sqrt{u}, \text{ so that } y=\int\frac{1}{2}\sqrt{u}du=\frac{1}{3}u^{\frac{3}{2}},$$

and therefore
$$\int x\sqrt{(x^2+1)}dx=y=\frac{1}{3}(x^2+1)^{\frac{3}{2}},$$

when u is replaced by its value in terms of x.

The process therefore consists in choosing a new variable u, calculating dy/du and then evaluating the u-integral, the variable u being finally replaced by its value in terms of the original variable.

We may now state the process in general terms. Given

$$y=\int F(x)dx, \text{ ....(1)}$$

so that
$$\frac{dy}{dx}=F(x).$$

Choose a new variable u, and from the equation between u and x calculate dx/du; then

$$\frac{dy}{du}=\frac{dy}{dx}\frac{dx}{du}=F(x)\frac{dx}{du}. \quad\text{.....................(2)}$$

Express $F(x)\ dx/du$ in terms of u alone, so that dy/du becomes a function of u; writing equation (2) in the integral form, we obtain

$$y=\int F(x)\frac{dx}{du}du. \quad\text{......................(3)}$$

From (1) and (3), the integrals in which are equal though differently expressed, we have

$$\int F(x)dx=\int F(x)\frac{dx}{du}du.$$

Hence the rule: **In the given integral replace** dx **by** $\frac{dx}{du}du$, **calculate** $\frac{dx}{du}$ **and express the new integrand,** $F(x)\frac{dx}{du}$, **in terms of** u **alone.**

It *may* be possible to evaluate the u-integral; when the evaluation has been effected, the variable must be replaced by its value in terms of the original one.

Example 1. Integrate $x/(x^2+1)$.

Write, $$y=\int\frac{x}{x^2+1}dx=\int\frac{x}{x^2+1}\frac{dx}{du}du.$$

Let $u=x^2+1$; therefore

$$\frac{du}{dx}=2x,\quad \frac{dx}{du}=\frac{1}{2x},\quad \frac{x}{x^2+1}\frac{dx}{du}=\frac{1}{2u}$$

and $$y=\int\frac{1}{2u}du=\tfrac{1}{2}\int\frac{du}{u}=\tfrac{1}{2}\log u=\tfrac{1}{2}\log(x^2+1).$$

In practice it is usually simpler to work with differentials, that is, to express dx in terms of du, as in the following example.

Example 2. Evaluate $\int\frac{(x-2)dx}{\sqrt{(2x^2-8x+5)}}$.

Let $u=2x^2-8x+5$; then

$$du=(4x-8)dx;\quad (x-2)dx=\tfrac{1}{4}du;$$

and $$\int\frac{(x-2)dx}{\sqrt{(2x^2-8x+5)}}=\tfrac{1}{4}\int\frac{du}{\sqrt{u}}=\tfrac{1}{2}\sqrt{u}=\tfrac{1}{2}\sqrt{(2x^2-8x+5)}.$$

Example 3. Integrate $x^2\sqrt{(x+2)}$.

Let $x+2=u^2$; then

$$dx=2udu\,;\ \sqrt{(x+2)}=u\,;\ x^2=u^4-4u^2+4$$

and

$$\int x^2\sqrt{(x+2)}dx=\int(u^4-4u^2+4)u\,.\,2udu$$
$$=2\int(u^6-4u^4+4u^2)du$$
$$=2(\tfrac{1}{7}u^7-\tfrac{4}{5}u^5+\tfrac{4}{3}u^3)$$
$$=\frac{2u^3}{105}(15u^4-84u^2+140).$$

Putting for u its value $(x+2)^{\frac{1}{2}}$ we find for the value of the integral

$$\tfrac{2}{105}(15x^2-24x+32)(x+2)^{\frac{3}{2}},$$

or

$$\tfrac{2}{105}(15x^3+6x^2-16x+64)\sqrt{(x+2)}.$$

The chief difficulty in these examples lies in the choice of the new variable; experience alone gives facility, but an attentive consideration of the substitutions suggested in Exercises V. 66-75 may help the beginner. Note, for instance, that in example 2 above $(x-2)dx$ is, but for a constant factor, the differential of $2x^2-8x+5$. The linear substitution indicated in Exercises VII., example 25 will frequently be useful.

EXERCISES. VII.

Integrate the functions 1–22.

1. $\dfrac{x}{x^2+a^2}$.
2. $\dfrac{x+a}{x^2+2ax+b^2}$.
3. $\dfrac{ax+b}{ax^2+2bx+c}$.
4. $x\sqrt{(3-x^2)}$.
5. $\dfrac{x}{\sqrt{(3-x^2)}}$.
6. $x\sqrt{(ax^2+b)}$.
7. $(x-a)\sqrt{(x^2-2ax+b^2)}$.
8. $\dfrac{ax+b}{\sqrt{(ax^2+2bx+c)}}$.
9. $x^2\sqrt{(x^3-1)}$.
10. $\dfrac{x}{(x^2+a^2)^2}$.
11. $\dfrac{2x-3}{(x^2-3x+8)^2}$.
12. $\dfrac{7-3x}{\sqrt{(3x^2-14x+10)}}$.
13. $x^2\sqrt{(x+1)}$.
14. $(2x+5)\sqrt{(x+3)}$.
15. $\dfrac{x^3}{\sqrt{(x-1)}}$.
16. $x^3\sqrt{(ax^4+b)}$.
17. $x^{n-1}(ax^n+b)^m$.
18. $[f(x)]^n f'(x)$.

19. $\dfrac{f'(x)}{f(x)}$. (This is the special case of example 18 when $n = -1$.)

20. $x^3\sqrt{(x^2+1)}$. **21.** $\dfrac{x}{(x+1)^{\frac{3}{2}}}$. **22.** $\dfrac{x}{(ax+b)^{\frac{3}{2}}}$.

23. Show that, if $u = \dfrac{1}{x^2}$, $\displaystyle\int \frac{dx}{(x^2+1)^{\frac{3}{2}}} = -\tfrac{1}{2}\int \frac{du}{(1+u)^{\frac{3}{2}}}$ and then evaluate the integral

$$\left[du = -\frac{2dx}{x^3};\ \int \frac{dx}{(x^2+1)^{\frac{3}{2}}} = \int \frac{dx}{x^3\left(1+\frac{1}{x^2}\right)^{\frac{3}{2}}} = \int \frac{-\frac{1}{2}du}{(1+u)^{\frac{3}{2}}}\right].$$

24. Integrate $1/(x^2+a^2)^{\frac{3}{2}}$ by the substitution $u = a^2/x^2$.

25. Integrate $1/(x^2+4x+5)^{\frac{3}{2}}$ by first putting $u = x+2$ and then $v = 1/u^2$.

[The substitution $u = x+2$ reduces x^2+4x+5 to u^2+1, in which the variable occurs only in the second power; the quadratic x^2+4x+5 becomes the *pure* quadratic u^2+1. Any quadratic can be reduced to a pure quadratic by **a linear substitution**; for

$$ax^2+bx+c = a\left(x+\frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} = au^2 + \frac{4ac-b^2}{4a}$$

if $u = x + b/2a$. This reduction of the quadratic is often useful.]

26. Integrate $1/(x^2+2ax+b)^{\frac{3}{2}}$.

[Use the linear substitution $u = x+a$.]

CHAPTER VI.

AREAS. DEFINITE INTEGRALS.

30. Areas. Newtonian Method. Let CPD (Fig. 13) be the graph of a function $F(x)$;

$$OA = a,\ AC = F(a)\ ;\ OM = x,\ MP = F(x).$$

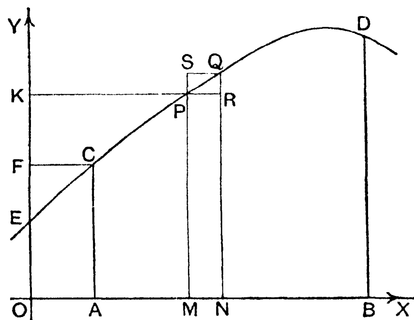


Fig. 13.

Let z denote the area $AMPC$; this area may be thought of as being generated by a variable ordinate, which sets out from the position AC and moves to the right. z is thus a function of x which is zero when $x = a$. To calculate z we first find dz/dx.

First, let every ordinate be positive. When x takes the increment $\delta x (= MN)$ the area z takes the increment $\delta z (=\text{area } MNQP)$. Draw PR, SQ parallel to MN. Then $MNQP$ is equal to the rectangle $MNRP$, together with the area PRQ which is less than the rectangle $PRQS$; therefore

$$\delta z = MP \times \delta x + (\text{a quantity less than } RQ \times \delta x),$$

$$\frac{\delta z}{\delta x} = MP + (\text{a quantity less than } RQ).$$

When δx converges to zero so does RQ, and therefore

$$\frac{dz}{dx}=MP=F(x)=\text{ordinate at } x. \qquad \text{(A)}$$

The differential dz is equal to $F(x)dx$, the rectangle $MNRP$.

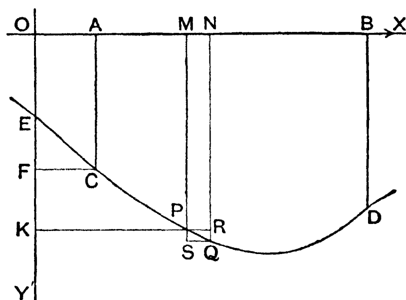


Fig. 14.

Secondly, suppose the ordinates to be negative (Fig. 14). The product $MP\delta x$ is now negative; we make the convention that, when the ordinates are negative and the increment δx positive, **the area shall be reckoned negative.** We then get the same value for dz/dx as before; but it must be remembered that the area is an **algebraical** quantity (see § 31, examples 1 and 2).

z is therefore that function of x whose derivative is $F(x)$, and which is equal to zero when $x=a$; in other words, z is that integral of $F(x)$ which is zero when $x=a$.

For definiteness, let $F(x)=4+7x-x^2$; then

$$z=\int(4+7x-x^2)dx+C=4x+\tfrac{7}{2}x^2-\tfrac{1}{3}x^3+C. \qquad (1)$$

Also, $0=4a+\frac{7}{2}a^2-\frac{1}{3}a^3+C$; $C=-(4a+\frac{7}{2}a^2-\frac{1}{3}a^3)$, and therefore

$$z=4x+\tfrac{7}{2}x^2-\tfrac{1}{3}x^3-(4a+\tfrac{7}{2}a^2-\tfrac{1}{3}a^3). \qquad (2)$$

If $OB=b$, the area $ABDC$ is the value of z when $x=b$; therefore the area $ABDC$ is equal to

$$4b+\tfrac{7}{2}b^2-\tfrac{1}{3}b^3-(4a+\tfrac{7}{2}a^2-\tfrac{1}{3}a^3). \qquad (3)$$

31. Definite Integrals. There is a special notation for the integral that is equal to the area $ABDC$, namely

$$\int_a^b (4+7x-x^2)\,dx\,; \qquad \qquad (4)$$

read, "the integral from a to b of $(4+7x-x^2)dx$."

The function denoted by the symbol (4) is called a **definite integral**; a and b are called the **limits** of the integral, a being the **lower** limit and b the **upper**. (The word "limit" here means merely "value of the variable of integration at one end of its range"; "end-value of the variable." This use of the word must not be confounded with the other technical meaning defined in § 16.) The distance or interval AB, equal to $b-a$, is called the **range** of integration.

By comparing equations (1) and (3), § 30, we see that the value of the definite integral (4) may be obtained by the rule: *Find the indefinite integral of* $4+7x-x^2$; *then replace* x *by* b, *the upper limit; next replace* x *by* a, *the lower limit; finally subtract the second result from the first.*

In finding the definite integral it is needless to add the constant C to the indefinite integral, because it disappears in the subtraction.

In general, if $f(x)$ is the indefinite integral of $F(x)$, that is, if $Df(x)=F(x)$ we have

$$\int_a^b F(x)\,dx=\Big[f(x)\Big]_a^b=f(b)-f(a). \qquad \qquad \text{(B)}$$

The function in brackets is the indefinite integral, and a, b on the right of the bracket are the limits; the symbol, taken as a whole, is a compact and convenient method of representing the rule for evaluating the integral.

The following examples should be carefully studied.

Example 1. Find the area between the graph of x^3-3x^2+2x, the x-axis, and the ordinates at $x=0$ and $x=2$.

The required area is

$$\int_0^2 (x^3-3x^2+2x)\,dx=\left[\frac{x^4}{4}-x^3+x^2\right]_0^2=0-0=0.$$

The reason for this apparently absurd result is, that from $x=0$ to $x=1$ the ordinates are positive, while from $x=1$ to $x=2$ they are negative. The first part of the area therefore is positive, and the

second part negative ; it happens that these two parts are *numerically* equal. The separate areas are

$$\int_0^1 (x^3 - 3x^2 + 2x)dx = \tfrac{1}{4} \ ; \quad \int_1^2 (x^3 - 3x^2 + 2x)dx = 0 - \tfrac{1}{4} = -\tfrac{1}{4}.$$

Before he tries to calculate an area, the student should **always sketch, however roughly, the graph of the function.**

Example 2. Show that $\int_b^a F(x)dx = -\int_a^b F(x)dx.$

By equation (B), if $F(x) = Df(x)$,

$$\int_b^a F(x)dx = \Big[f(x)\Big]_b^a = f(a) - f(b) = -\Big[f(x)\Big]_a^b = -\int_a^b F(x)dx.$$

Hence, **we may interchange the limits of an integral if at the same time we change the sign of the integral.**

Geometrically interpreted, this result means that the area traced out by an ordinate moving from the position $x = b$ to the position $x = a$ is of **opposite sign** to the area traced out when the motion is from $x = a$ to $x = b$. It is very important to observe that the *sign* of an integral depends not only on the sign of the integrand $F(x)$ but also on the relative (algebraic) magnitude of the limits a and b ; even if $F(x)$ is positive the integral will be negative when the lower limit is greater than the upper. Thus,

$$\int_2^1 x^2 dx = \left[\frac{x^3}{3}\right]_2^1 = \tfrac{1}{3} - \tfrac{8}{3} = -\tfrac{7}{3}.$$

Example 3. Prove $\int_a^b F(x)dx = \int_a^c F(x)dx + \int_c^b F(x)dx.$

The area represented by the integral on the left is the sum of the areas represented by the integrals on the right, so that the equation is correct.

Example 4. What is the value of a definite integral when the two limits are equal ?

The value is zero, because the ordinate does not move and no area is generated.

Example 5. Show that $\int_a^b F(x)dx = \int_a^b F(u)du.$

Each integral represents the same area, because the graph is the same whether the abscissa is denoted by x or by u.

Otherwise, if $F(x) = D_x f(x)$ then $F(u) = D_u f(u)$ and each integral is equal to $f(b) - f(a)$. Hence,

A definite integral is a function of its **limits** and **not of the variable of integration.**

The area $AMPC$ (Fig. 13) may be denoted by either of the integrals

$$z = \int_a^{OM} F(x)dx \ \text{ or } \ z = \int_a^{OM} F(u)du.$$

Here OM is the abscissa of the final position of the generating ordinate. We may, if we please, put x for OM but *this* x is not the same as the x in $F(x)dx$; as will be seen later (§ 33) $F(x)dx$ is rather a *type* of a series of terms than a single term.

By § 30 (A), $dz/dx = F(x)$; hence

$$\frac{d}{dx}\int_a^x F(x)dx = F(x),\ \frac{d}{dx}\int_a^x F(u)du = F(x).$$

Example 6. Evaluate $\int_2^6 (x+1)\sqrt{(x-2)}dx$ by the substitution $x-2=u^2$.

$(x+1)\sqrt{(x-2)}dx$, when expressed in terms of u, becomes $(2u^4+6u^2)du$.

Now, what range of values does u take as x increases from the lower limit 2 to the upper limit 6? When $x=2$ the equation $x-2=u^2$ gives $u=0$, and when $x=6$ the same equation gives $u=2$; so that, as x increases from 2 to 6, u increases from 0 to 2. The lower limit for the new integral is therefore 0 and the upper 2. Hence

$$\int_2^6 (x+1)\sqrt{(x-2)}dx = \int_0^2 (2u^4+6u^2)du = \left[\frac{2u^5}{5}+2u^3\right]_0^2 = 28\tfrac{4}{5}.$$

Note. When changing the variable in a *definite* integral, it is not necessary to replace the new variable in the result by its value in terms of the old, *provided* we choose the limits of the new integral, as we have done, to *correspond* to the limits of the old; the new lower corresponds to the old lower and the new upper to the old upper.

The following example is important.

Example 7. Prove $\int\sqrt{(a^2-x^2)}dx = \frac{1}{2}x\sqrt{(a^2-x^2)} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$.

In Fig. 15 OA, OB are two perpendicular radii of a circle of radius a; $OM=x$, $MP=\sqrt{(a^2-x^2)}$, the angle $BOP=\theta$ radians and $\sin\theta = x/a$.

Now,

$$\text{area } OMPB = \triangle OMP + \text{sector } BOP$$
$$= \tfrac{1}{2}x\sqrt{(a^2-x^2)} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$$

because

$$\text{sector } BOP = \tfrac{1}{2}a^2\theta \text{ and } \theta = \sin^{-1}(x/a).$$

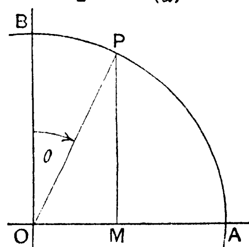


Fig. 15.

But the area is also represented by the integral

$$\int_0^x \sqrt{(a^2-x^2)}dx;$$

therefore $$\int_0^x \sqrt{(a^2-x^2)}dx = \tfrac{1}{2}x\sqrt{(a^2-x^2)} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right).$$

By example 5, the derivative of this integral is $\sqrt{(a^2-x^2)}$, so that the *indefinite* integral $\int\sqrt{(a^2-x^2)}dx$ has the value stated.

32. Interpretation of Area. The interpretation of the area z will depend on the nature of the quantities represented by the abscissa and the ordinate. The interpretation is most readily made by considering dz, which is equal to $MP \times dx$; of course, z and dz represent the same kind of quantity. The following are typical cases.

If OM or x represents time and MP or $F(x)$ velocity, then dz represents *velocity* $\times$ *time*, or *distance*; the area $AMPC$ represents the distance gone in the time represented by AM.

If OM represents time and MP acceleration, then dz represents *acceleration* $\times$ *time*, or *velocity*; the area $AMPC$ represents the velocity gained in the time represented by AM.

If OM represents the distance a force moves its point of application and MP represents the force (the direction of the force being constant and always that in which its point of application moves), then dz represents *force* $\times$ *distance*, that is, *the work done by the force*; the area $AMPC$ represents the work done by the force in moving its point of application through the distance represented by AM.

As regards scales, if 1 inch for abscissae represents, say, 5 seconds and 1 inch for ordinates a velocity of 10 feet per second, then 1 square inch of area will represent a distance of 10×5 or 50 feet. If 1 inch for abscissae represents 10 feet and 1 inch for ordinates a force of 50 lbs., then 1 square inch of area will represent 10×50 or 500 foot-pounds of work; and so on.

If $F(x)$ is the derivative of $f(x)$ the area $AMPC$ is $f(x)-f(a)$; in other words, the *area* between the graph of $F(x)$, the x-axis, the fixed ordinate $F(a)$ and the variable ordinate $F(x)$ is equal to the variable *ordinate* $f(x)$ diminished by the fixed ordinate $f(a)$. The increase of the area is thus equal to the increase of the ordinate $f(x)$. If the fixed ordinate $F(a)$ is so chosen that $f(a)=0$ (for example, if

$$F(x)=A+Bx+Cx^2, \quad f(x)=Ax+\tfrac{1}{2}Bx^2+\tfrac{1}{3}Cx^3,$$

and $a=0$, then $f(a)=0$) the area is equal to the ordinate $f(x)$; in any case, the area only differs from the ordinate $f(x)$ by a constant. The graph of $f(x)$ is called, from this

consideration, **the integral curve of the graph of F(x).** (See Chapter XIV.)

EXERCISES. VIII.

Calculate the definite integrals 1–25.

1. $\int_1^5 dx.$ 2. $\int_1^5 (2x+3)dx.$ 3. $\int_{-3}^2 (3-x)dx.$ 4. $\int_{-4}^{-1} x^2dx.$

5. $\int_{-4}^{-1} x^3dx.$ 6. $\int_0^5 (6+8x-3x^2)dx.$ 7. $\int_{-5}^5 (6+8x-3x^2)dx.$

8. $\int_{-1}^1 (x-x^3)dx.$ 9. $\int_1^3 \frac{dx}{x^2}.$ 10. $\int_1^3 \frac{dx}{x^3}.$

11. $\int_1^3 \frac{dx}{x^4}.$ 12. $\int_{-1\cdot 5}^6 (2x+3)^2dx.$ 13. $\int_{-1}^4 \frac{dx}{(2x+3)^2}.$

14. $\int_0^8 \sqrt{x}dx.$ 15. $\int_{-2}^6 \sqrt{(x+2)}dx.$ 16. $\int_1^4 \frac{dx}{\sqrt{x}}.$

17. $\int_0^4 \frac{dx}{\sqrt{(2x+1)}}.$ 18. $\int_0^9 \frac{dx}{\sqrt[3]{(7x+1)}}.$ 19. $\int_{\frac{1}{2}}^2 \frac{dx}{x}.$

20. $\int_{a^2}^{b^2} \frac{dx}{x}.$ 21. $\int_0^3 \frac{dx}{4-x}.$ 22. $\int_{-1}^1 \frac{dx}{2x+3}.$

23. $\int_3^5 \frac{dx}{x^2-4}.$ 24. $\int_{-1}^1 \frac{dx}{4-x^2}.$ 25. $\int_a^b \frac{dx}{x^{1\cdot 4}}.$

Evaluate by appropriate substitutions the integrals in Examples 26–34.

26. $\int_0^4 x\sqrt{(x^2+9)}dx.$ 27. $\int_0^2 x\sqrt{(4-x^2)}dx.$ 28. $\int_0^a x\sqrt{(a^2-x^2)}dx.$

29. $\int_0^a \frac{xdx}{\sqrt{(a^2-x^2)}}.$ 30. $\int_0^b \frac{xdx}{a^2+x^2}.$ 31. $\int_1^3 \frac{x^3dx}{\sqrt{(x-1)}}.$

32. $\int_0^8 \frac{xdx}{(x+1)^{\frac{3}{2}}}.$ 33. $\int_0^4 \frac{dx}{(x^2+9)^{\frac{3}{2}}}.$ 34. $\int_1^5 (x^2+2x-1)\sqrt{(x-1)}dx.$

35. Find the area bounded by the parabola $y=5+6x-x^2$, the positive parts of the coordinate axes and the ordinate at $x=5$.

36. Trace the curve $y=x-x^3$ from $x=0$ to $x=1$, and find the area between it and the x-axis.

37. Trace the curve $y=x^2-x^3$ from $x=-1$ to $x=1$, and find the area between the curve, the x-axis and the ordinates at $x=-1$ and $x=1$.

38. Find the area between the parabola $y=ax^2+bx+c$, the x-axis and the ordinates at $x=-h$ and $x=h$.

If y has the values y_1, y_2, y_3 respectively when x has the values $-h$, 0, h calculate the values of a, b, c in terms of y_1, y_2, y_3, h, and show that the area is $\frac{1}{3}h(y_1+y_3+4y_2)$.

39. If y_1, y_2, y_3 have the same meaning as in example 38, and if $y=ax^3+bx^2+cx+g$, show that the area is $\frac{1}{3}h(y_1+y_3+4y_2)$.

[Note that the area is $\frac{2h}{3}(bh^2+3g)$, so that only the values of b and g need be calculated.]

40. Show that the area bounded by the parabola $y^2=4ax$ and the double ordinate through the point (b, c) on it is $\frac{4}{3}bc$.

41. Find the area between the hyperbola $xy=c^2$, the x-axis and the ordinates at $x=a$ and $x=b$, assuming a and b to be both positive and a less than b.

42. Find the area between the curve $y=k/x^n$, the x-axis and the ordinates at $x=x_1$ and $x=x_2$, assuming that n is greater than 1, and that x_1, x_2 are both positive and x_1 less than x_2.

If y_1, y_2 are the values of y when x has the values x_1, x_2 respectively, show that the area is $(x_1y_1-x_2y_2)/(n-1)$.

43. Find the area between the curve $y=x^n/c^{n-1}$, the x-axis and the ordinates at $x=x_1$ and $x=x_2$, assuming that n is greater than 1, and that x_1, x_2 are both positive and x_1 less than x_2.

If y has the values y_1, y_2 respectively when x has the values x_1, x_2, show that the area is $(x_2y_2-x_1y_1)/(n+1)$.

44. Find the area between the curve $y=c^3/x^2$, the x-axis and the ordinates at $x=1$ and $x=b$ $(b>1)$. To what value does the area tend when b becomes infinite?

45. Find the area of the ellipse $x^2/a^2+y^2/b^2=1$.

[Use the integral proved in § 31, example 7.]

46. Prove

$$\int\sqrt{(2ax-x^2)}\,dx=\tfrac{1}{2}(x-a)\sqrt{(2ax-x^2)}+\tfrac{1}{2}a^2\sin^{-1}\left(\frac{x-a}{a}\right).$$

[Put $u=x-a$ and then use § 31, example 7.

Or, take $\sqrt{(2ax-x^2)}$ as the ordinate of a circle, referred to a diameter and the tangent at one end of it as coordinate axes.]

47. Evaluate the following integrals:

(i) $\int_1^3\sqrt{(-3+4x-x^2)}\,dx$; (ii) $\int_1^3 x\sqrt{(-3+4x-x^2)}\,dx$;

(iii) $\int_a^b\sqrt{\{-ab+(a+b)x-x^2\}}\,dx$; (iv) $\int_a^b x\sqrt{\{-ab+(a+b)x-x^2\}}\,dx$.

48. Deduce from § 31, example 7, that

$$\frac{d}{dx}\sin^{-1}\left(\frac{x}{a}\right)=\frac{1}{\sqrt{(a^2-x^2)}},\quad \int\frac{dx}{\sqrt{(a^2-x^2)}}=\sin^{-1}\left(\frac{x}{a}\right).$$

49. Show that the area bounded by a curve, the y-axis and the lines $y=a$, $y=b$, parallel to the x-axis, is given by

$$\int_a^b x\,dy.$$

[If z denote the area $FCPK$ (Figs. 13, 14) prove $dz/dy=KP=x$.]

50. Find the area between the parabola $y^2=4ax$, the y-axis and the line $y=h$.

CHAPTER VII.

INTEGRAL AS LIMIT OF A SUM. SIMPSON'S RULES.

33. The Definite Integral as the Limit of a Sum. There is another point of view from which the definite integral may be considered; to illustrate it, we take the following problem.

A straight line AB (Fig. 16), of length a, is divided into n equal parts and each part is multiplied by the square of the distance from A of that end of the part which is *nearest* A; find the sum of the n products, and the limit of the sum when n becomes infinite.

Let MN be one of the parts, M being the r^{th} point of division; then $AM = ra/n$, $MN = a/n$ and $AM^2 \,.\, MN$, the term of the sum arising from the part MN, is $\left(\frac{ra}{n}\right)^2 . \frac{a}{n}$.

The sum required, S say, is therefore

$$S = 0^2 \times \frac{a}{n} + \left(\frac{a}{n}\right)^2 . \frac{a}{n} + \left(\frac{2a}{n}\right)^2 . \frac{a}{n} + \dots$$

$$+ \left(\frac{ra}{n}\right)^2 . \frac{a}{n} + \dots + \left\{\frac{(n-1)a}{n}\right\}^2 . \frac{a}{n}. \quad \dots\dots\dots\dots\dots (1)$$

The sum (1) is written more compactly thus:

$$S = \sum_{r=0}^{r=n-1} \left(\frac{ra}{n}\right)^2 . \frac{a}{n}. \quad \dots\dots\dots\dots\dots (2)$$

The symbol Σ, read "sigma," is the Greek form of capital S; the whole symbol, read "sum of $\left(\frac{ra}{n}\right)^2 . \frac{a}{n}$ from $r=0$ to $r=n-1$," is to be interpreted as meaning that we are to make r equal successively to $0, 1, 2, \dots n-1$ in the expression $\left(\frac{ra}{n}\right)^2 . \frac{a}{n}$ and then add the terms so obtained.

It is proved in any text-book of Algebra (and is easily verified) that

$$1^2 + 2^2 + 3^2 + \dots + (n-1)^2 = \tfrac{1}{6}(n-1)n(2n-1).$$

But $$S=\{1^2+2^2+3^2+\ldots+(n-1)^2\}\times\frac{a^3}{n^3},$$

so that $$S=\tfrac{1}{6}(n-1)n(2n-1)\times\frac{a^3}{n^3}=\frac{a^3}{3}\left(1-\frac{3}{2n}+\frac{1}{2n^2}\right),$$

and therefore the limit of S when n becomes ∞ is $\frac{1}{3}a^3$.

Suppose now that each part, as MN, is multiplied by the square of the distance from A of that end of the part which is *furthest from* A, as AN^2; we find instead of (2)

$$S'=\sum_{r=1}^{r=n}\left(\frac{ra}{n}\right)^2\cdot\frac{a}{n}=\tfrac{1}{6}n(n+1)(2n+1)\times\frac{a^3}{n^3}\ \ldots\ldots\ldots\ldots\ldots\ldots\ (3)$$

and the limit of S' when n becomes ∞ is $\frac{1}{3}a^3$, the same value as before.

It is worth noting that we shall obtain *the same limit* if we take, for each part, the distance from A of *any point* in that part; because if, for example, L is any point between M and N the quantity AL^2 will lie between AM^2 and AN^2, and the sum will therefore lie between S and S'. Since S and S' have the same limit, the new sum will also have that limit.

We shall now obtain the above limits in a different way.

Let $AM=x$, $AN=x+\delta x$; then $\delta x=MN=a/n$, and (2), (3) become

$$S=\sum_{x=0}^{x=a-a/n}x^2\delta x\ \ldots\ldots\ldots\ldots\ (4);\quad S'=\sum_{x=a/n}^{x=a}x^2\delta x.\ \ldots\ldots\ldots\ldots\ldots\ (5)$$

It is to be understood that the values to be given to x in (4) and (5) are the distances from A of the successive points of division; in (4) the first value of x is 0 and the last $AA_{n-1}=a-a/n$, while in (5) the first value of x is $AA_1=a/n$ and the last $AB=a$. In each term $\delta x=a/n$.

Now draw, with A as origin and AB as x-axis, the parabola $y=x^2$, and at the points of division of AB raise the ordinates. Complete, for each part into which AB is divided, the two rectangles, such as $MNRP$, $MNQS$ (Fig. 16).

We now interpret the sums (2), (3) or their equivalents (4), (5) *geometrically*.

The sum (4) is the sum of the *inner* rectangles, such as $MNRP$; because when $x=AM$ we have $x^2=MP$ and $x^2\delta x=$ rect. $MNRP$, and a similar interpretation holds for every term in (4).

The sum (5) on the other hand is the sum of the *outer* rectangles, such as $MNQS$; because in this case a part such as MN is multiplied by AN^2, and therefore, for the part MN, $x^2\delta x=AN^2 . MN=NQ . MN=$ rect. $MNQS$.

Obviously the sum of the inner rectangles is less, and the sum of the outer rectangles greater, than the area $ABDP$ bounded by the curve, the x-axis and the ordinate BD. Hence
$$S < \text{area } ABDP < S'. \quad\quad (6)$$

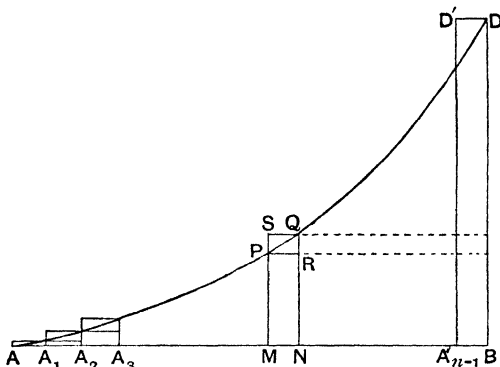


Fig. 16.

Now, the difference $S'-S$ is simply the sum of the rectangles such as $PRQS$; this sum is, as may be seen by sliding each small rectangle horizontally into the rectangle $A_{n-1}BDD'$, equal to that rectangle. But rect. $A_{n-1}BDD'$ is equal to a^3/n, because $BD=a^2$ and $A_{n-1}B=a/n$; so that $S'-S=a^3/n$. Hence when n becomes infinite, *the limit* of $S'-S$ is zero.

Again, the differences, area $ABDP-S$ and $S'-$area $ABDP$, are each less than $S'-S$, so that when n becomes infinite *the limit* of each of these differences is zero; in other words, the limit both of S and of S' is the area $ABDP$.

Therefore, *expressing the area as an integral*,

$$\underset{n=\infty}{\mathrm{L}} \sum_{x=0}^{x=a-a/n} x^2\delta x = \int_0^a x^2 dx = \underset{n=\infty}{\mathrm{L}} \sum_{x=a/n}^{x=a} x^2\delta x. \quad\quad (7)$$

Since the value of the integral is $\frac{1}{3}a^3$, we find the same result as before.

As before, it is plain that, *so far as the limit is concerned*, we may suppose x in the typical term $x^2\delta x$ to be the distance from A of *any* point in MN.

The definite integral thus appears as the limit of a certain sum, and this mode of viewing it is of great importance in various applications, as we shall see. The passage from the sum to the integral has been effected by considering the factor $(ra/n)^2$ (or x^2) of the term $\left(\frac{ra}{n}\right)^2 \cdot \frac{a}{n}$ (or $x^2\delta x$) as the ordinate of a curve, and the term itself as the area of a rectangle which is approximately equal to a strip of the area between the curve and the x-axis. Since we suppose all the functions we deal with to be representable by a graph, the method applies to all cases.

By considerations that are essentially identical with those adduced in this particular case (a full discussion will be found in the author's *Calculus*, § 131) we can arrive at the general theorem which may be stated thus.

In Fig. 13, § 30, let $MP = F(x)$, $MN = \delta x$, then

$$\text{area } ABDC = \sum_{x=a}^{x=b} F(x)\delta x \text{ approximately} \quad \ldots\ldots\ldots\ldots (8)$$

$$= \sum_{x=a}^{x=b} F(x+\delta x)\delta x \text{ approximately} \quad \ldots\ldots (9)$$

$$= \int_a^b F(x)\,dx \text{ exactly.}$$

In this case, the statements $x=a$, $x=b$ attached to Σ are meant to indicate the values of x for the two bounding ordinates; this meaning is slightly different from that adopted above, but is convenient.

In (8) the typical term $F(x)\delta x$ is $MP \cdot MN$, the area of $MNRP$, while in (9) the typical term $F(x+\delta x)\delta x$ is $NQ \cdot MN$, the area of $MNQS$; *so far as the limit is concerned*, the rectangle corresponding to the strip $MNQP$ may have for its altitude MP or NQ or any ordinate between MP and NQ. The Σ indicates for (8) the sum of the inner rectangles and for (9) the sum of the outer.

The origin of the symbol $\int \ldots dx$ for integration will now be obvious; $\int$ is a form of the letter "s" and dx is the representative of δx.

The method of defining an integral as the limit of a sum is due to Leibniz; the method followed in §§ 30, 31 is that of Newton.

34. Volumes. In Fig. 13, § 30, let the curve CPD make a complete revolution about the x-axis; the curve thus traces out a surface. The section of the surface by any plane perpendicular to OX, **the axis of revolution**, is a circle. We wish to find the volume intercepted between the surface and the planes through A and B perpendicular to OX.

Let V be the volume intercepted between the planes through A and M perpendicular to OX, and let δV be the increment of volume for the increment MN or δx of x. Then δV is greater than the cylinder traced out by rect. $MNRP$ but less than that traced out by rect. $MNQS$;

therefore $\delta V > \pi MP^2 . \delta x$ but $\delta V < \pi NQ^2 . \delta x$,

so that $\delta V/\delta x$ is greater than πMP^2 but less than πNQ^2.

Hence
$$\frac{dV}{dx} = \pi MP^2 = \pi(\text{ordinate at } x)^2. \quad \text{............(1)}$$

The volume required is the integral of πMP^2 from a to b, just as in § 31 the area is the integral of MP from a to b.

We may also use the method of § 33. The volume clearly lies between
$$\sum_{x=a}^{x=b} \pi MP^2 \delta x \text{ and } \sum_{x=a}^{x=b} \pi NQ^2 \delta x;$$
but the limit of these sums is the same for both, namely
$$\int_a^b \pi y^2 dx. \quad \text{..............................(2)}$$

As a rule we need only consider one of the inequalities for δV and proceed thus:
$$\text{vol.} = \sum_{x=a}^{x=b} \pi MP^2 . \delta x \text{ approximately}; \ \text{vol.} = \int_a^b \pi MP^2 dx.$$

Example. Find the volume of a sphere of radius R.

The equation of the generating semi-circle is
$$MP^2 = y^2 = R^2 - x^2,$$
and therefore the volume is
$$\int_{-R}^{R} \pi (R^2 - x^2) dx = \left[\pi\left(R^2 x - \frac{x^3}{3}\right)\right]_{-R}^{R} = \frac{4\pi}{3} R^3.$$

Examples will be found in Exercises IX.

35. Arcs of Curves. Areas of Curved Surfaces. When the distance between two points P, Q on a curve is small, the arc PQ and the chord PQ are nearly equal. We assume as an axiom, that when Q tends to P as its limiting position the quotient (arc $PQ \div$ chord PQ) tends to unity as its limit.

Let the arc CP (Fig. 17), measured from some point C, be denoted by s and let δs or arc PQ be the increment of s due to the increment δx or MN of x. Then

$$(\text{chord } PQ)^2 = PR^2 + RQ^2 = (\delta x)^2 + (\delta y)^2. \quad \text{.........(1)}$$

The limit of (chord $PQ/\delta x$) is the same as the limit of $\delta s/\delta x$; because

$$\frac{\text{chord } PQ}{\delta x} = \frac{\text{chord } PQ}{\text{arc } PQ} \cdot \frac{\delta s}{\delta x}$$

and the limit of (chord $PQ \div$ arc PQ) is unity. Divide (1) by $(\delta x)^2$ and take the limit; therefore

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2 \text{ or } ds^2 = dx^2 + dy^2. \quad \text{............(2)}$$

When $dx = MN$ or PR, then if NQ meets the tangent at P at T, $dy = RT$, and

$$PT^2 = PR^2 + RT^2 = dx^2 + dy^2.$$

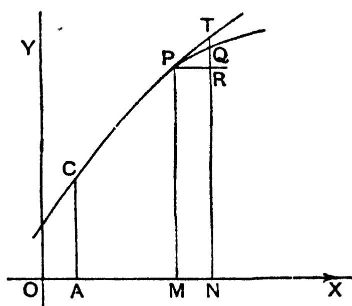


Fig. 17.

The second equation in (2) shows that $ds = PT$. Thus, **so far as the limit is concerned, the chord PQ, the arc PQ and the portion PT of the tangent can be substituted for each other.**

If PT makes the angle ϕ with the x-axis, then $\phi = \angle RPT$, and we have the values

$$\tan\phi = \frac{dy}{dx},\ \cos\phi = \frac{dx}{ds},\ \sin\phi = \frac{dy}{ds}. \qquad \text{(3)}$$

If P is the position at time t of a point moving along the curve CP, its velocity v is

$$v = \underset{\delta t=0}{\mathrm{L}} \frac{\delta s}{\delta t} = \frac{ds}{dt}. \qquad \text{(4)}$$

The component velocities are

$$\frac{dx}{dt} = \frac{ds}{dt}\cos\phi,\ \frac{dy}{dt} = \frac{ds}{dt}\sin\phi. \qquad \text{(5)}$$

s is given by the integral

$$s = \int \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}\,dx\,; \qquad \text{(6)}$$

but there are very few curves whose length can be expressed by means of the ordinary functions.

Let S denote the superficial area of the surface traced out by the revolution of the arc CP about OX (Fig. 17), and let δS be the increment of S due to the increment δx or MN; δS is the area traced out by the arc PQ, but in finding the limit of $\delta S/\delta x$ we may take either the chord PQ or PT instead of the arc PQ.

The area of the surface of the conical frustum traced out by PQ is $\pi(MP+NQ) . PQ$; therefore

$$\frac{\delta S}{\delta x} = \pi(MP+NQ) . \frac{PQ}{\delta x} = \pi(2MP+RQ)\frac{\delta s}{\delta x},$$

$$\frac{dS}{dx} = 2\pi MP . \frac{ds}{dx} \text{ or } dS = 2\pi MP . ds. \qquad \text{(7)}$$

Note that $dS = 2\pi y ds$, not $2\pi y dx$.

Example. Find the area of a spherical cap of height h.

If the radius of the sphere is R the ordinate of the generating circle is

$$y = MP = \sqrt{(R^2 - x^2)}.$$

$$\frac{dy}{dx} = \frac{-x}{\sqrt{(R^2-x^2)}}\,; \quad \frac{ds}{dx} = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}} = \frac{R}{\sqrt{R^2-x^2}}\,;$$

$$\frac{dS}{dx} = 2\pi MP\frac{ds}{dx} = 2\pi R.$$

$$S = \int_{R-h}^{R} 2\pi R dx = 2\pi Rh.$$

The area is thus equal to the curved surface of a circular cylinder of height h and of radius equal to that of the sphere.

Examples will be found in Exercises IX., 14–17.

36. Simpson's Rules. In many cases a function is given by its graph and not by an analytical expression; even when the expression is known it is often impossible to find the indefinite integral of the function. We shall investigate a method for calculating an approximate value of the integral of the function when a limited number of the ordinates of the graph is known; the integral is represented as an area.

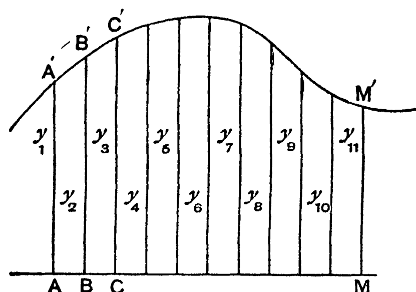


Fig. 18.

In Fig. 18 let $y_1, y_2, y_3 \dots$ be the values of the ordinates $AA', BB', CC', \dots$ respectively, and suppose the distances AB, BC, ... between consecutive ordinates to be *equal*, each distance being equal to h.

Now, we can determine the constants a, b, c in the equation

$$y = ax^2 + bx + c, \qquad \dots\dots\dots\dots(1)$$

so that the parabola represented by (1) shall pass through any three given points. We shall therefore assume that the three points A', B', C' lie on a parabola; if the points are not very far apart the assumption is usually near enough to the truth.

Let B be taken as origin, and BC, BB' as axes of x and y respectively; then A', B', C' are the points $(-h, y_1)$, $(0, y_2)$, (h, y_3). The abscissa of A' is $-h$ because BA is negative.

The equation (1) is referred to the axes just chosen, and we apply it to calculate the area $ACC'A'$.

$$\text{Area } ACC'A' = \int_{-h}^{h}(ax^2+bx+c)dx = \frac{2h}{3}(ah^2+3c). \text{ ....(2)}$$

Since the points A', B', C' lie on the curve given by (1), we have

$$y_1 = ah^2 - bh + c,\ y_2 = c,\ y_3 = ah^2 + bh + c,$$

and therefore, solving for c and ah^2 (the value of b is not needed), we find $c = y_2$, $ah^2 = \frac{1}{2}(y_1 + y_3 - 2y_2)$. Inserting these values in (2) and reducing, we obtain

$$\text{area } ACC'A' = \tfrac{1}{3}h(y_1 + y_3 + 4y_2). \text{ ....(3)}$$

Equation (3) expresses the area in terms of y_1, y_2, y_3, h, so that the area can be calculated when these four quantities are known.

If the equation of the curve $A'B'C'$ is actually, and not merely approximately, of the form (1), then the value (3) is *exact*.

The student may prove, precisely as has been done for equation (1), that if the assumed equation is

$$y = ax^3 + bx^2 + cx + d$$

we get the same expression (3) for the area.

We can now generalise the result (3). Suppose the area $AMM'A'$ to be divided into an even number, $2n$, of strips by an odd number, $2n+1$, of equidistant ordinates. The formula (3) may be applied in succession to the n double strips. If S is the sum of the n expressions we find,

$$S = \tfrac{1}{3}h(y_1 + y_3 + 4y_2) + \tfrac{1}{3}h(y_3 + y_5 + 4y_4) + \dots + \tfrac{1}{3}h(y_{2n-1} + y_{2n+1} + 4y_{2n})$$

$$= \tfrac{1}{3}h\{y_1 + y_{2n+1} + 2(y_3 + y_5 + \dots + y_{2n-1}) + 4(y_2 + y_4 + \dots + y_{2n})\}. \text{ ...(4)}$$

Formula (4) is known as **Simpson's Rule**; it may be stated thus:

Let the area be divided into an even number of strips by equidistant ordinates; find (i) the sum of the extreme ordinates, (ii) twice the sum of the other odd ordinates, (iii) four times the sum of the even ordinates; add the three sums thus obtained and multiply this total sum by **one-third** of the common distance between the ordinates.

Though the rule has been proved for areas it is of course applicable to any definite integral, because the integrand can always be represented by the ordinate of a curve. An important case is that of the mensuration of solids; in this case y_1, y_2, ... are the areas of equidistant sections.

Equation (3) includes many of the most important formulae of mensuration; in Exercises IX. several of these are stated.

Example. Calculate $\int_1^2 \frac{dx}{x}$.

Let $2n+1=11$, $h=0{\cdot}1$, $y=1/x$. Then $y_1, y_2, y_3 \dots y_{11}$ are the values of y when x has the values 1, 1·1, 1·2, ... 2 respectively. An easy calculation gives

$$y_1+y_{11}=1{\cdot}5,\ y_3+y_5+y_7+y_9=2{\cdot}7281746,$$

$$y_2+y_4+y_6+y_8+y_{10}=3{\cdot}4595394,\ S=0{\cdot}693150.$$

The exact value of the integral is $\log_e 2$, that is, 0·693147.

EXERCISES. IX.

1. The volume of a right circular cone is one-third of the base multiplied by the height. (Compare examples 3, 4.)

2. If the area, S, of a section of a surface by a plane perpendicular to the x-axis is a function of x show, by the same reasoning as in § 34, that the volume between two planes perpendicular to the x-axis is $\int S\,dx$, the integral being taken between proper limits.

3. Show that the volume of any cone is one-third of the base multiplied by the height.

[If h is the height, A the base and S the section distant x from the vertex, then $S : A = x^2 : h^2$.]

4. If the areas of the ends of the frustum of any cone are A and B, and the height of the frustum h, show that the volume of the frustum is

$$\tfrac{1}{3}h\{A+\sqrt{(AB)}+B\}.$$

[Complete the cone and let the height of the cone needed for this be h_1; then if $h_2=h+h_1=$ height of completed cone, we have

$$A : h_1^2 = B : h_2^2 = \lambda \text{ say};\ A=\lambda h_1^2,\ B=\lambda h_2^2;$$

$$\text{vol. of frustum} = \tfrac{1}{3}(\lambda h_2^3 - \lambda h_1^3) = \tfrac{1}{3}(h_2-h_1)(\lambda h_1^2+\lambda h_1 h_2+\lambda h_2^2),$$

which gives the result, since $h=h_2-h_1$, $A=\lambda h_1^2$, $B=\lambda h_2^2$.]

5. A reservoir has plane sloping sides and ends; the top and bottom are horizontal rectangles of sides a, b and a', b' respectively and the depth is h. Show that the volume is

$$\tfrac{1}{6}h\{ab+a'b'+(a+a')(b+b')\}.$$

6. The volume of a spherical cap of height h is $\pi h^2(R-\frac{1}{3}h)$, the radius of the sphere being R.

[The volume is $\int_{R-h}^{R}\pi(R^2-x^2)dx$; see example of § 34.]

7. A **prolate spheroid** is the surface generated by an ellipse which revolves about its major axis; an **oblate spheroid** is the surface generated by an ellipse which revolves about its minor axis. If the major and minor axes are $2a$ and $2b$ respectively show that the volume of the prolate spheroid is $\frac{4}{3}\pi ab^2$ and of the oblate $\frac{4}{3}\pi a^2b$.

8. The **tore** or **anchor-ring** is the surface generated by a circle which revolves about an axis in its plane, the axis not intersecting the circle (though it may be a tangent to it). If a is the radius of the circle and c the distance of its centre from the axis, show that the volume of the tore is $2\pi^2a^2c$.

[Let the equation of the circle be $x^2+(y-c)^2=a^2$, the x-axis being the axis of revolution and the y-axis passing through the centre of the circle. If $OM=x$ and if the perpendicular to the x-axis from M cut the circle at P_1 and P_2, then

$$MP_1=y_1=c-\surd(a^2-x^2),\quad MP_2=y_2=c+\surd(a^2-x^2),$$

$$\text{vol.}=2\int_0^a\pi(y_2{}^2-y_1{}^2)dx=8\pi c\int_0^a\surd(a^2-x^2)\,dx.$$

The integral is the area of a quadrant of a circle and therefore equal to $\frac{1}{4}\pi a^2$.]

9. The volume intercepted between the plane through $x=h$, perpendicular to the x-axis, and the **paraboloid** generated by the revolution of the parabola $y^2=4ax$ about the x-axis is $2\pi ah^2$.

10. If, in example 2, $S=ax^2+bx+c$ and if the values of S are S_1, S_2, S_3 respectively when x has the values $-h$, 0, h, show that the volume V bounded by the surface and the sections S_1, S_3 is given by

$$V=\tfrac{1}{3}h(S_1+S_3+4S_2).$$

[This is the form of Simpson's Rule for a solid corresponding to that in § 36, equation (3), for an area.]

11. Apply example 10, that is Simpson's Rule, to obtain the values in examples 1, 3, 4, 5, 6, 7, 9.

12. If d_1 is the head diameter, d_2 the bung diameter, and h the depth of a cask, show that when the curve of the cask is a parabola the volume is

$$\frac{\pi}{12}h\{d_1{}^2+2d_2{}^2-\tfrac{4}{10}(d_2-d_1)^2\}.$$

When the upper and lower halves of the cask are equal frustums of a paraboloid of revolution, the greatest bases being joined in the middle of the cask, show that the volume is

$$\frac{\pi}{8}h(d_1{}^2+d_2{}^2).$$

[A paraboloid of revolution is the surface generated by the revolution of a parabola about its axis; see example 9.]

13. If $y=ax^3+bx^2+cx+g$, and if the values of y are y_1, y_2, y_3, y_4 respectively when x has the values $-\frac{3}{2}h$, $-\frac{1}{2}h$, $\frac{1}{2}h$, $\frac{3}{2}h$, show that the area bounded by the graph of y, the x-axis and the extreme ordinates y_1, y_4 is
$$\tfrac{3}{8}h\{y_1+y_4+3(y_2+y_3)\}.$$
[In this case the area is divided into 3 strips by equidistant ordinates, the common distance being h; the result may be extended as in § 36. The rule obtained from this result is known as **Simpson's Second Rule**; but it is not much used.]

14. Show that for the **semi-cubical parabola** $ay^2=x^3$
$$\frac{ds}{dx}=\sqrt{\left(1+\frac{9x}{4a}\right)},\quad s=\frac{8a}{27}\left(1+\frac{9x}{4a}\right)^{\frac{3}{2}}-\frac{8a}{27},$$
the arc s being measured from the origin.

15. Show that for the parabola $y^2=4ax$,
$$\frac{ds}{dx}=\sqrt{\left(1+\frac{a}{x}\right)},\quad \frac{ds}{dy}=\sqrt{\left(1+\frac{y^2}{4a^2}\right)};$$
and that the surface of the portion of the paraboloid of example 9 up to the plane there mentioned is
$$\int_0^h 4\pi\sqrt{a}\sqrt{(x+a)}\,dx=\frac{8\pi}{3}\{a^{\frac{1}{2}}(a+h)^{\frac{3}{2}}-a^2\}.$$

16. Show that for the parabola $y=x^2/p$
$$\frac{ds}{dx}=\sqrt{\left(1+\frac{4x^2}{p^2}\right)}=1+\frac{2x^2}{p^2}\text{ approx.,}$$
$$s=x+\frac{2x^3}{3p^2}\text{ approx.,}$$
the arc being measured from the vertex.

Show also that the length of the arc from the vertex to the point (b, c) is approximately
$$b+\tfrac{2}{3}\frac{c^2}{b}.$$

[Since (b, c) is on the parabola, $c=b^2/p$ or $p=b^2/c$; putting b for x and b^2/c for p in the value found for s, we get the result. This approximation to the arc is often used in mechanics.]

17. Show that for the ellipse $x^2/a^2+y^2/b^2=1$, e being the eccentricity, so that $b=a\sqrt{(1-e^2)}$,
$$\frac{ds}{dx}=\frac{\sqrt{(a^2-e^2x^2)}}{\sqrt{(a^2-x^2)}}$$
and that the surface of the prolate spheroid (example 7) is
$$2\int_0^a 2\pi y\frac{ds}{dx}\,dx=4\pi\sqrt{(1-e^2)}\int_0^a\sqrt{(a^2-e^2x^2)}\,dx$$
$$=2\pi a^2\left\{1-e^2+\sqrt{(1-e^2)}\frac{\sin^{-1}e}{e}\right\}.$$

[Note that $$\int_0^a\sqrt{(a^2-e^2x^2)}\,dx=e\int_0^a\sqrt{\left(\frac{a^2}{e^2}-x^2\right)}\,dx$$

and apply example 7, § 31, where a has to be replaced by a/e.]

CHAPTER VIII.

APPLICATIONS TO MECHANICS.

37. Centroids. First Moments. We now consider some applications to Mechanics and begin with centroids.

It is shown in works on Mechanics that the x-coordinate, $\bar{x}$ say, of the centroid of n particles of masses $m_1, m_2, \dots m_n$ situated at points whose x-coordinates are $x_1, x_2, \dots x_n$ respectively, is given by the equation

$$\bar{x}=\frac{m_1x_1+m_2x_2+\dots+m_nx_n}{m_1+m_2+\dots+m_n}=\frac{\Sigma mx}{\Sigma m}.$$

There are similar expressions for $\bar{y}$ and $\bar{z}$, the y- and the z-coordinates of the centroid.

The centroid of a volume, area or line is the centroid of a mass of unit density occupying the volume, area or line.

Other names in common use for the centroid are: **centre of inertia, centre of gravity.** The sum Σmx is often called the **first moment** of the masses.

To see how integrals replace sums, take a very simple case.

Example 1. Find the centroid of a thin straight rod of uniform density and thickness.

Let the length AB (Fig. 19) of the rod be a and the mass of unit length λ; then the total mass M of the rod is λa.

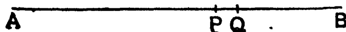


Fig. 19.

Divide AB into n equal parts, of which PQ is one. Let $AP=x$, $PQ=\delta x$; then the mass of PQ is $\lambda\delta x$.

If we suppose the mass of PQ to be concentrated at P, the moment of the mass of PQ about A will be $x \, . \, \lambda\delta x$; while if we suppose the mass of PQ to be concentrated at Q, the moment about A will be $(x+\delta x)\lambda\delta x$. We may assume the moment to be greater than $x \, . \, \lambda\delta x$ but less than $(x+\delta x)\lambda\delta x$.

If $\bar{x}$ is the distance of the centroid from A the moment of the whole mass M about A is $M\bar{x}$. Hence

$$M\bar{x} > \sum_{x=0}^{x=a} x\lambda\delta x, \text{ but } M\bar{x} < \sum_{x=0}^{x=a} (x+\delta x)\lambda\delta x.$$

But when the number n of the parts, such as PQ, into which the rod is divided, becomes infinite each of these two sums converges to the same limit (§ 33), namely the integral from 0 to a of $x\lambda dx$; therefore

$$M\bar{x} = \int_0^a x\lambda dx = \tfrac{1}{2}\lambda a^2 = \tfrac{1}{2}Ma,$$

so that $\bar{x} = \frac{1}{2}a$.

In this and similar cases it is sufficient to consider only one of the inequalities and to write

$$M\bar{x} = \Sigma x\lambda\delta x \text{ approx.}; \quad M\bar{x} = \int_0^a x\lambda dx.$$

In practice the differential dx is frequently used instead of the increment δx; we shall however adhere meantime to the notation of increments.

Example 2. Find the centroid of a thin plate (or *lamina*) of uniform density and thickness, the plate having the shape of a quadrant of a circle.

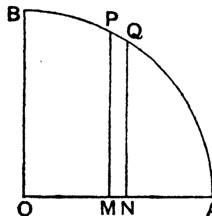


Fig. 20.

Let a be the radius of the plate and σ the mass of unit area; then the mass M of the plate is $\frac{1}{4}\pi\sigma a^2$.

Divide the plate into narrow strips like $MNQP$ (Fig. 20); we may suppose the mass of the strip to be concentrated at the middle point of MP.

Let $OM = x$, $MN = \delta x$, $MP = y = \surd(a^2 - x^2)$. The mass of the strip may be taken as $\sigma y\delta x$ and the moment of this mass about OA is $\frac{1}{2}y \, . \, \sigma y\delta x$ or $\frac{1}{2}\sigma y^2\delta x$. Hence $\bar{y}$ is given by

$$M\bar{y} = \sum_{x=0}^{x=a} \tfrac{1}{2}\sigma y^2\delta x \text{ approx.}; \quad M\bar{y} = \int_0^a \tfrac{1}{2}\sigma y^2 dx.$$

But $y^2 = a^2 - x^2$, and therefore by integration

$$M\bar{y} = \left[\tfrac{1}{2}\sigma(a^2x - \tfrac{1}{3}x^3)\right]_0^a = \tfrac{1}{3}\sigma a^3 = \tfrac{1}{4}\pi\sigma a^2 \, . \, \frac{4a}{3\pi},$$

so that

$$\bar{y} = \frac{4a}{3\pi}$$

since $M = \frac{1}{4}\pi\sigma a^2$.

In the same way, or from symmetry, we see that $\bar{x}=4a/3\pi$.

The student may show that the integral is the same if we suppose the elementary mass to be $\sigma NQ.MN$, concentrated at the middle point of NQ.

Example 3. Prove the following theorems :

(i) If an arc of a plane curve revolves about an axis in its plane which does not intersect it, the surface generated by the arc is equal to the length of the arc multiplied by the length of the path of the centroid of the arc.

(ii) If a plane area revolves about an axis in its plane which does not intersect it, the volume generated by the area is equal to the area multiplied by the length of the path of the centroid of the area.

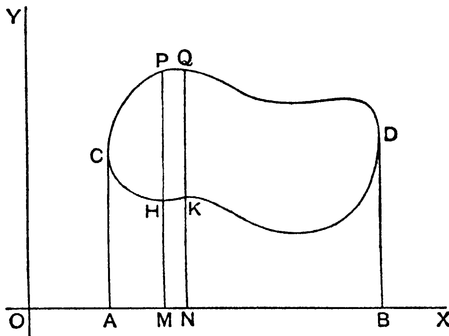


Fig. 21.

Take OX (Fig. 21) as the axis of revolution and let C be the point on the closed curve $CPDH$ nearest OY, and D the point furthest from OY. Denote by y_1 any ordinate MP of the arc CPD, by y_2 any ordinate MH of the arc CHD, by s_1 the arc CP and by s_2 the arc CH, δs_1 and δs_2 being the increments PQ and HK respectively of these arcs. The lengths of the arcs CPD, CHD may be denoted by l_1, l_2 respectively.

The ordinates $\bar{y}_1$, $\bar{y}_2$, $\bar{y}_3$ of the centroids of the arcs CPD, CHD and the closed curve $CPDH$ are given by the equations

$$l_1\bar{y}_1=\int_0^{l_1} y_1 ds_1,\quad l_2\bar{y}_2=\int_0^{l_2} y_2 ds_2,\quad (l_1+l_2)\bar{y}_3=\int_0^{l_1} y_1 ds_1+\int_0^{l_2} y_2 ds_2.$$

But the area of the surface generated by the arc CPD is (§ 35)

$$S_1=2\pi\int_0^{l_1} y_1 ds_1$$

which by the first of the above integrals is equal to $2\pi\bar{y}_1 l_1$, that is, to the distance $2\pi\bar{y}_1$ travelled by the centroid of CPD multiplied by l_1 the length of the arc CPD.

Similarly the theorem is proved for the arc CHD and for the closed curve $CPDH$.

Next, to find the ordinate $\bar{y}$ of the centroid of the area bounded by the curve $CPDH$, we may take the strip $HKQP$ as the element of area and suppose it concentrated at the middle point of HP. The area of the strip is $(y_1-y_2)\delta x$ approximately, where $OM=x$, $MN=\delta x$, and the moment of the strip about OX is

$$\tfrac{1}{2}(y_1+y_2)(y_1-y_2)\delta x, \text{ that is, } \tfrac{1}{2}(y_1^2-y_2^2)\delta x.$$

Therefore $$\bar{y}\times \text{area } CPDH=\int_{OA}^{OB}\tfrac{1}{2}(y_1^2-y_2^2)dx.$$

But the volume generated by the area is equal to the integral multiplied by 2π (§ 34), that is, equal to $2\pi\bar{y}\times$ area $CPDH$.

These theorems are usually known as the **Theorems of Pappus.**

Example 4. If the centroid of a plane area S lies on the x-axis, show that $\int y dS$ taken over the area is zero.

If the area is divided into n small portions, of which δS may be taken as a type, and if y is the ordinate of any point in δS, then the moment of δS about the x-axis is $y\,\delta S$ approximately and the moment of the area S is $\Sigma y\,\delta S$ approximately, where the summation Σ includes all the n portions like δS. The moment of the area is the limit of $\Sigma y\,\delta S$ when n becomes infinite and each of the portions converges at the same time to zero. This limit is denoted by the integral $\int y dS$.

But if $\bar{y}$ is the ordinate of the centroid of the area we have

$$S\bar{y}=\int y dS.$$

If the centroid lies on the x-axis then $\bar{y}=0$, and therefore the integral is zero.

Conversely, when $\int y dS$ is zero the centroid lies on the x-axis.

Similarly, if we take δV as an *element* of volume the y-coordinate of the centroid of the volume V is given by

$$V\bar{y}=\int y dV,$$

where the integration extends throughout the volume V.

As a rule double integration (§ 70) is required for the evaluation of $\int y dS$, but in simple cases double integration can be avoided. Thus, in example 2, $\delta S=y\,\delta x$ and only simple integration is needed; similarly, in example 3, $\delta S=(y_1-y_2)\delta x$.

38. Centres of Pressure. The intensity of pressure (§ 25, example 2) at the depth x feet, in a heavy liquid of uniform density ρ pounds per cubic foot, is $(p_0+\rho x)$ pounds per square foot, p_0 being the intensity of pressure at the free

surface. In our examples p_0 will be neglected, since the effect of p_0 can be easily estimated without integration; p_0 contributes to the thrust on any plane area, S square feet, the amount $p_0 S$ pounds, this force acting at the centroid of the area.

The thrust on one face of a plane area, S square feet in extent, placed horizontally at the depth x feet, is simply $\rho x S$ pounds acting at the centroid of the area. If the area is not horizontal the intensity of the pressure is different at different depths. To find the thrust on the area S in this case divide the area into n small portions, the area of one portion being δS. If the depth of any point in δS below the free surface is x feet the thrust on δS is approximately $\rho x \delta S$ pounds, and the thrust on the whole area S is approximately $\Sigma \rho x \delta S$, the summation Σ including all the n elements δS. The limit of $\Sigma \rho x \delta S$, when n becomes infinite and each element δS converges to zero (that is, the integral $\int \rho x dS$) is the thrust on the area S.

To find the point in the area, called the **centre of pressure,** at which this thrust, or resultant pressure-force, acts take moments about two lines in the plane of the area as in the following examples.

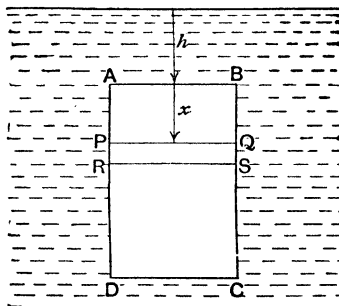


Fig. 22.

Example 1. Find the centre of pressure of a rectangle whose plane is vertical, one side being parallel to the free surface.

Let $ABCD$ (Fig. 22) be the rectangle; $AB=a$, $AD=b$; h the depth of AB below the surface, to which AB is parallel.

Divide the rectangle into narrow strips by lines parallel to AB, and let $AP=x$, $PR=\delta x$. The area $PQSR=\delta S=a\,\delta x$; the thrust on $PQSR$ is $\rho(h+x)a\,\delta x$ approximately, and therefore the thrust P on the area $ABCD$ is

$$P=\int_0^b \rho(h+x)a\,dx=\rho abh+\tfrac{1}{2}\rho ab^2.$$

Let $\bar{x}$ be the depth below AB of the centre of pressure and take moments about AB. The moment about AB of the thrust on $PQSR$ is $x\,.\,\rho(h+x)a\delta x$, and therefore the moment about AB of the thrust on $ABCD$ is

$$\int_0^b x\,.\,\rho(h+x)a\,dx, \quad \text{or} \quad \rho a\int_0^b (hx+x^2)\,dx,$$

which is equal to $\qquad \tfrac{1}{2}\rho ab^2h+\tfrac{1}{3}\rho ab^3.$

But this is equal to $P\times\bar{x}$, the moment of P about AB. Hence we find

$$\bar{x}=\frac{\tfrac{1}{2}\rho ab^2h+\tfrac{1}{3}\rho ab^3}{\rho abh+\tfrac{1}{2}\rho ab^2}=\frac{bh+\tfrac{2}{3}b^2}{2h+b}.$$

If AB is in the surface of the liquid, $h=0$ and $\bar{x}=\tfrac{2}{3}b$.

It is obvious that the centre of pressure also lies on the line through the middle point of AB parallel to AD.

Example 2. Find the centre of pressure of a triangle ABC, the base BC being in the surface of the liquid and the plane of the triangle being inclined to the vertical at the angle θ.

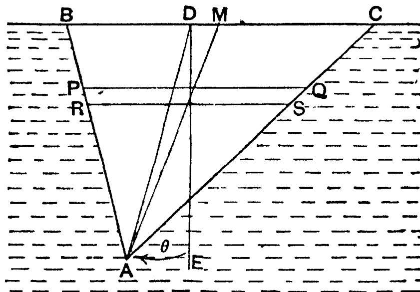


Fig. 23.

Let M (Fig. 23) be the middle point of BC; it is easy to see that the centre of pressure lies in AM. Let AD be perpendicular to BC, PQ and RS parallel to BC, DE vertical, $\angle EDA=\theta$.

$BC=a$, $DA=h$; the perpendicular distance from BC to PQ is x and from PQ to RS is δx.

$$PQ=\frac{a}{h}(h-x)\,; \quad \text{area of } PQSR=\frac{a}{h}(h-x)\delta x \text{ approx.}$$

The intensity of pressure along PQ is $\rho x \cos\theta$ and the thrust on the strip $PQSR$ may be taken to be

$$\rho x \cos\theta \times \frac{a}{h}(h-x)\delta x, \quad \text{or} \quad \frac{\rho a \cos\theta}{h}(hx-x^2)\delta x.$$

Hence the thrust P on the triangle ABC is

$$P = \frac{\rho a \cos\theta}{h}\int_0^h (hx - x^2)\,dx = \tfrac{1}{6}\rho a h^2 \cos\theta.$$

Next, take moments about BC; the moment of the thrust on $PQSR$ is

$$x \,.\, \frac{\rho a \cos\theta}{h}(hx - x^2)\,\delta x,$$

and therefore the moment of the thrust on the triangle ABC is

$$\frac{\rho a \cos\theta}{h}\int_0^h (hx^2 - x^3)\,dx = \tfrac{1}{12}\rho a h^3 \cos\theta.$$

The distance $\bar{x}$ from BC of the centre of pressure is therefore

$$\bar{x} = \tfrac{1}{12}\rho a h^3 \cos\theta \div \tfrac{1}{6}\rho a h^2 \cos\theta = \tfrac{1}{2}h.$$

The centre of pressure is therefore the middle point of AM; its position is independent of the inclination θ.

39. Moments of Inertia or Second Moments. If $r_1, r_2, \ldots r_n$ are the distances from an axis OR of n particles of masses $m_1, m_2, \ldots m_n$ respectively, the sum

$$m_1 r_1^2 + m_2 r_2^2 + \ldots + m_n r_n^2 = \Sigma m r^2 \quad \ldots\ldots\ldots\ldots\ldots(1)$$

is called the **moment of inertia** of the set of particles about OR.

The moment of inertia of a continuous body is obtained by supposing it to be divided into a large number n of small pieces and finding the limit, for n becoming infinite while each of the n pieces converges to zero, of the moment of inertia of the n pieces. The sum (1) is then replaced by an integral. If $\delta m_1, \delta m_2, \ldots$ are the masses of the small pieces and $r_1, r_2, \ldots$ the distances from the axis of any point in the masses $\delta m_1, \delta m_2, \ldots$ respectively, the moment of inertia about the axis is the limit of $\Sigma r^2 \delta m$ and is given by the integral

$$\int r^2 dm$$

taken between proper limits.

If the total mass of the system is M, and if k is chosen sc that

$$Mk^2 = \Sigma m r^2, \text{ or } Mk^2 = \int r^2 dm,$$

the quantity k, which is of the nature of a length, is called the **radius of gyration** of the system about the axis.

We shall suppose the bodies treated to be of uniform density. When the density is taken equal to unity the masses may be treated as volumes, areas, or lines; in this case the moment of inertia is often called the **second moment** of the volume, area, or line about the axis.

The symbol I is generally used to denote a moment of inertia; if M is the mass of the system and k the radius of gyration about an axis, then $I = Mk^2$ for that axis.

The work of finding moments of inertia is greatly simplified by the following theorems; the proofs are very simple (see any text-book of Mechanics).

Theorem 1. If OX, OY, OZ are three rectangular axes, and if I_x, I_y, I_z are the moments of inertia about OX, OY, OZ respectively of a plane lamina lying in the plane XOY, then

$$I_z = I_x + I_y.$$

Theorem 2. If I_R is the moment of inertia about any axis OR, I_G the moment of inertia about a parallel axis through the centre of inertia G, a the distance between the two axes, and M the total mass of the system, then

$$I_R = I_G + Ma^2.$$

Example 1. A thin straight rod of uniform density about an axis through one end perpendicular to the rod.

Let AB (Fig. 19, p. 89) be the rod, and take the same notation as in § 37, example 1. The mass of PQ is $\lambda\delta x$ and the moment of inertia of this element may be taken to be $x^2 . \lambda\delta x$. Hence for the moment of inertia I of the rod we have

$$I = \int_0^a x^2 \lambda\, dx = \tfrac{1}{3}\lambda a^3 = \tfrac{1}{3}Ma^2,$$

so that $k^2 = \frac{1}{3}a^2$, $k = 0{\cdot}577a$.

The moment about an axis through the middle point of the rod perpendicular to the rod is $\frac{1}{12}Ma^2$, as may be proved either by integration or by Theorem 2.

Example 2. A uniform rectangular lamina about an axis through its centre parallel to one side.

Let the lamina be $ABCD$ (Fig. 22, p. 93) and let the axis be parallel to AD; $AB = a$, $AD = b$. Divide the lamina into narrow strips like $PQSR$; if δm is the mass of $PQSR$ the moment of this mass is, by example 1, $\frac{1}{12}a^2\delta m$. The moment of the whole mass M is therefore $\frac{1}{12}a^2 . M$.

Similarly, the moment about an axis through its centre parallel to

AB is $\frac{1}{12}b^2 . M$, and therefore the moment about an axis through its centre perpendicular to its plane is, by Theorem 1,

$$\tfrac{1}{12}(a^2+b^2)M.$$

The moment of a uniform rectangular parallelepiped (a brick or cuboid) whose edges are a, b, c about an axis through its centre parallel to an edge, the edge c say, is

$$\tfrac{1}{12}(a^2+b^2)M,$$

where M is the total mass of the parallelepiped. To prove this result, divide the parallelepiped into thin slices by planes perpendicular to the edge c. The moment of a slice is

$$\tfrac{1}{12}(a^2+b^2)\times(\text{mass of slice}),$$

so that the required moment is $\frac{1}{12}(a^2+b^2)M$.

Example 3. A uniform circular lamina of radius a about a diameter.

Let $X'OX$ be the diameter, $Y'OY$ the perpendicular diameter and I_x, I_y, I_z the moments about OX, OY, OZ respectively, where OZ is perpendicular to the plane of the lamina. It is clear, from symmetry, that $I_y=I_x$, and therefore (Theorem 1) $I_x=\frac{1}{2}I_z$.

To find I_z, divide the lamina into narrow concentric strips. The mass δm of the strip bounded by circles of radii x and $x+\delta x$ may be taken as $2\pi\sigma x\delta x$, where σ is the mass of unit area of the lamina. The distance from OZ of each point of the strip may be taken as x, so that the moment of δm is $x^2 . 2\pi\sigma x\,\delta x$ or $2\pi\sigma x^3\delta x$. Hence

$$I_z=\int_0^a 2\pi\sigma x^3 dx=\tfrac{1}{2}\pi\sigma a^4=\tfrac{1}{2}Ma^2$$

where M is the mass of the lamina. Therefore $I_x=\frac{1}{4}Ma^2$ and $k=\frac{1}{2}a$.

Example 4. A sphere of uniform density about a diameter.

Divide the sphere into thin slices by planes perpendicular to the diameter. The mass δm of a slice distant x from the centre of the sphere is $\pi\rho(a^2-x^2)\delta x$, where ρ is the density and a the radius of the sphere, and δx the thickness of the slice. The moment of this circular lamina about the diameter of the sphere is, by example 3, $\frac{1}{2}(a^2-x^2)\delta m$, that is $\frac{1}{2}\pi\rho(a^2-x^2)^2\delta x$. Hence for the moment of inertia I we have

$$I=\tfrac{1}{2}\pi\rho\int_{-a}^{a}(a^2-x^2)^2dx=\tfrac{1}{2}\pi\rho . \tfrac{16}{15}a^5=\tfrac{2}{5}a^2M$$

where $M=\frac{4}{3}\pi\rho a^3$, the mass of the sphere.

Example 5. Bending moment.

In the usual theory of the bending of beams the intensity of stress at any point of a section of the beam, made by a plane perpendicular to its length, is of the form Ex/R where R is the radius of curvature of the curve into which the beam is bent, E is a constant, and x is the distance of the point from a line in the section called the neutral axis. The stress across a small section δA containing the point is $Ex\delta A/R$; this stress may be resolved into a force $Ex\delta A/R$ acting at a point in

the neutral axis, and a couple $Ex^2\delta A/R$. Summing these over the area A of the section we find as the resultant of the stresses:

$$\text{(i) a force, } \frac{E}{R}\Sigma x\delta A\text{ ; (ii) a couple, } \frac{E}{R}\Sigma x^2\delta A.$$

For *pure flexure* the force is zero, and therefore $\Sigma x\delta A$ is zero; the centroid of the section will in this case lie in the neutral axis (§ 37, example 4). The stress thus reduces to a couple EI/R, where $I(=\Sigma x^2\delta A)$ is the second moment of the area A about the neutral axis. This couple is equal numerically to the **bending moment** of the applied forces. EI is called the **flexural rigidity** of the beam.

40. Work done by an Expanding Gas. Suppose the gas to be confined in a long rigid cylinder closed at one end and fitted with a piston which is free to slide, the cross section of the cylinder being constant, equal to S square feet. Let the intensity of pressure be p pounds per square foot.

The work, δW say, done by the gas in pushing out the piston a small distance δx feet is, approximately, $pS\delta x$ foot-pounds. But $S\delta x$ is equal to δv, the increment of the volume v. Hence $\delta W = p\delta v$ approximately; taking the limit for δv converging to zero we find

$$\frac{dW}{dv} = p, \quad \text{or} \quad dW = pdv. \quad\text{................}(1)$$

The work done in increasing the volume from v_1 to v_2 is, in foot-pounds,

$$W = \int_{v_1}^{v_2} pdv. \quad\text{........................}(2)$$

If the gas is compressed from volume v_2 to volume v_1 the integral (2) gives the work required to produce the compression.

Equations (1) and (2) hold whatever be the form of the containing vessel, the walls being flexible in whole or in part; but the proof need not be given here. The work done is represented by the area between the graph of p, the v-axis, and the ordinates to the v-axis at v_1 and v_2.

The three equations connecting p, v, and the *absolute* temperature τ are.

$$\text{(i) } \frac{pv}{\tau} = C_1\text{; (ii) } pv = C_2\text{; (iii) } pv^{\gamma} = C_3.$$

Equation (ii) holds for *isothermal* expansion, equation (iii) for *adiabatic* expansion. The constants C_1, C_2, C_3 are different from one another.

For air at 32° F. or 493° absolute temperature, v being the volume in cubic feet of one pound of air and p the pressure in pounds per square foot, we have

$$C_1 = 53{\cdot}18, \quad C_2 = 26220, \quad \gamma = 1{\cdot}404.$$

These values are taken from Prof. Ewing's treatise on the steam-engine*; the absolute temperature is obtained by adding 461 to the Fahrenheit temperature.

Example 1. One pound of dry air at volume v_1 and (absolute) temperature τ_1 expands adiabatically to volume v_2; find the work done and the change in temperature.

By equation (2), since $pv^\gamma = C_3$, we have

$$W = \int_{v_1}^{v_2} \frac{C_3}{v^\gamma} dv = \frac{1}{\gamma - 1}\left(\frac{C_3}{v_1^{\gamma-1}} - \frac{C_3}{v_2^{\gamma-1}}\right). \quad \text{.................. (3)}$$

If we put $p_1 v_1^\gamma$ for C_3 in each of the fractions in (3) we get

$$W = \frac{p_1 v_1}{\gamma - 1}\left\{1 - \left(\frac{v_1}{v_2}\right)^{\gamma-1}\right\} = \frac{p_1 v_1}{\gamma - 1}\left\{1 - \frac{1}{R^{\gamma-1}}\right\} \quad \text{........... (4)}$$

where $R = v_2/v_1$, the ratio of expansion.

If in the first of the fractions in (3) we put $p_1 v_1^\gamma$ for C_3, and in the second $p_2 v_2^\gamma$ for C_3 we find

$$W = \frac{1}{\gamma - 1}\left(\frac{p_1 v_1^\gamma}{v_1^{\gamma-1}} - \frac{p_2 v_2^\gamma}{v_2^{\gamma-1}}\right) = \frac{p_1 v_1 - p_2 v_2}{\gamma - 1}. \quad \text{................. (5)}$$

If τ_2 is the temperature at volume v_2 we have $p_2 v_2 = C_1 \tau_2$, by (i); also $p_1 v_1 = C_1 \tau_1$ and therefore by (5)

$$W = \frac{C_1(\tau_1 - \tau_2)}{\gamma - 1}. \quad \text{................................ (6)}$$

Suppose $\tau_1 = 661$ (200° F.) and $v_2 = 2v_1$; then from (4) and (6), since $p_1 v_1 = C_1 \tau_1$, we have

$$W = 21240, \quad \tau_1 - \tau_2 = 161.$$

The work done is therefore 21240 foot-pounds, and the temperature has fallen 161°; the temperature has fallen from 200° F. to 39° F.

If the air at 39° F. were compressed adiabatically to half the volume the work required to produce the compression would be 21240 foot-pounds, and the temperature would rise to 200° F.

* *The Steam-Engine and other Heat-Engines.* By J. A. Ewing. (Cambridge: University Press.)

Example 2. If the expansion described in example 1 is isothermal, find the work done.

In this case $pv = C_2$ and equation (2) gives

$$W = \int_{v_1}^{v_2} \frac{C_2}{v} dv = C_2 \log_e\left(\frac{v_2}{v_1}\right) = p_1 v_1 \log_e R. \quad \text{.................}(7)$$

Suppose $\tau_1 = 500(39^\circ \text{ F.})$, $v_2 = 2v_1$ so that $R = 2$; then

$$W = 53{\cdot}18 \times 500 \times \log_e 2 = 53{\cdot}18 \times 500 \times 0{\cdot}6931 = 18430.$$

The work done in this case is therefore 18430 foot-pounds.

EXERCISES. X.

1. Show that the coordinates of the centroid of a plane lamina of uniform density in the shape of a quadrant of an ellipse whose axes are $2a$, $2b$ are given by

$$\bar{x} = 4a/3\pi, \quad \bar{y} = 4b/3\pi.$$

2. Find the centroid of a uniform right circular cone the area of whose base is A and whose height is h.

[The centroid clearly lies on the axis. Take a section perpendicular to the axis at the distance x from the vertex ; the area of the section is x^2A/h^2, and the mass of the slice of thickness δx is $\rho x^2 A\,\delta x/h^2$, ρ being the density. Take moments about an axis through the vertex parallel to the section ; then

$$M\bar{x} = \int_0^h x \cdot \frac{\rho x^2 A}{h^2} dx = \tfrac{1}{4}\rho h^2 A \; ; \quad \bar{x} = \tfrac{3}{4}h.]$$

3. Show that the centroid of the area bounded by an arc of the parabola $y^2 = ax$, the x-axis and the ordinate at the point (h, k) is given by

$$\bar{x} = \tfrac{3}{5}h, \quad \bar{y} = \tfrac{3}{8}k.$$

4. ACB is a semi-circle of radius R ; O is the middle point of the diameter AB, and C is the middle point of the arc AB. Show by the theorems of Pappus that the centroids of the arc and the area lie on OC at the distances from O given by

$$\bar{y} = \frac{2R}{\pi} \text{ for the arc ; } \quad \bar{y} = \frac{4R}{3\pi} \text{ for the area.}$$

5. Show that the surface of the tore (Exercises IX., 8) is $4\pi^2 ac$, and that the volume is $2\pi^2 a^2 c$.

6. A triangle ABC has its vertex A in the surface of a liquid and has its base horizontal ; M is the middle point of the base BC. Show that its centre of pressure is at H where $AH = \tfrac{3}{4}AM$.

7. One of the parallel sides AB of a trapezium $ABCD$ is in the surface of a liquid; if $AB = a$, $DC = b$, and if the perpendicular distance between AB and DC is h, show that the centre of pressure lies on the line joining the middle points of AB and DC, and that its perpendicular distance from AB is

$$\frac{a + 3b}{a + 2b} \cdot \frac{h}{2}.$$

Deduce from this result the position of the centre of pressure (i) of a parallelogram with one side in the surface, (ii) of a triangle with its base in the surface, (iii) of the triangle of example 6.

8. In example 1 of § 38 show that, when h is large compared with b, the centre of pressure all but coincides with the centroid of the rectangle.

9. If in example 1 of § 38 the depth of the centroid of the rectangle $ABCD$ is h_1 and the depth of the centre of pressure y, and if the rectangle is lowered so that the new depths of the centroid and the centre of pressure are h_2 and z respectively, show that

$$z = \frac{h_2^2 - h_1^2 + h_1 y}{h_2}.$$

10. A plane area S is immersed vertically in a liquid so that the depth of the centroid G of the area is h_1; show that the depth y of the centre of pressure is given by

$$h_1 S y = h_1^2 S + I,$$

where I is the second moment of the area about the horizontal through G.

Apply the result to show that the equation of example 9 holds for any plane area, z and h_2 having the same meaning as in that example.

Find the moments of inertia in the cases given in examples 11–19, the density being supposed uniform and the mass being M in each case.

11. A circular lamina of radius a, about a tangent.

12. A circular lamina of radius a, about an axis through a point on its circumference perpendicular to its plane.

13. A sphere of radius a, about a tangent line.

14. A sphere of radius a, about an axis distant c from its centre.

15. A rectangular lamina of sides a, b, about an axis through the middle point of the side a perpendicular to its plane.

16. A right circular cylinder whose height is h and whose cross section has a radius a, (i) about the axis of the cylinder, (ii) about a diameter of one of its circular ends.

17. A hollow circular cylinder, about the common axis of the two bounding surfaces, the inner and outer radii of a cross section being a and b respectively.

18. A right circular cone whose height is h and whose base has a radius a, (i) about its axis, (ii) about an axis through its vertex perpendicular to the axis of the cone.

19. A triangular lamina of height h, (i) about its base, (ii) about an axis through its vertex parallel to its base.

20. Show that the flexural rigidity (the stiffness) of a beam of rectangular cross section is proportional to the product of the breadth and the cube of the depth.

21. The pressure of one pound of saturated steam at 347° F. is 130 lb. per sq. in. and the volume is 3·44 cub. ft. Find the work done in an adiabatic expansion that doubles the volume, and the fall in temperature. ($\gamma = 1{\cdot}135$.)

22. Find the work done in suddenly compressing one pound of dry air, originally at 32° F., to three-quarters of its original volume; state also the temperature immediately after compression.

23. One pound of dry air at volume v_1, pressure p_1 and (absolute) temperature τ_1, is subjected to the following process: (i) it expands isothermally to volume v_2, taking in heat and doing work; (ii) it expands adiabatically from volume v_2, doing work at the expense of its internal energy, till its volume has become v_3 and its temperature τ_2; (iii) it is compressed isothermally at temperature τ_2 from volume v_3, work being spent on the gas (or *negative* work being done by the gas) till its volume has become v_4, v_4 being on the same adiabatic as v_1; (iv) it is compressed adiabatically till its volume has become v_1. Find the total work done by the gas, the work done during compression being considered negative; show that the work is represented by the area bounded by the two isothermals and the two adiabatics.

CHAPTER IX.

DIFFERENTIATION OF DIRECT TRIGONOMETRIC FUNCTIONS.

41. Trigonometric Limit. When an angle is small, say less than 5°, the sine of the angle is approximately equal to the number of radians in the angle. Even for an angle of 10°, the error in taking the number of radians for the sine is just a little greater than $\frac{1}{2}$ per cent. For,

$$10 \text{ degrees} = 0{\cdot}1745 \text{ radians}, \ \sin 10° = 0{\cdot}1736,$$

$$\text{percentage error} = \frac{0{\cdot}0009}{0{\cdot}1736} \times 100 = 0{\cdot}52.$$

In the language of limits, this fact is expressed as follows: if θ is the number of radians in an angle

$$\underset{\theta=0}{\mathrm{L}} \frac{\sin\theta}{\theta} = 1. \qquad \text{......(1)}$$

We shall show that this limit is equivalent to another limit already assumed (§ 35), namely that the limit of the quotient (chord ÷ arc) is unity when the arc converges to zero.

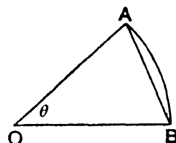


Fig. 24.

Let the radius OA (Fig. 24) of the circular arc AB be r, and let the angle AOB be θ radians; then,

$$\text{chord } AB = 2r\sin\tfrac{1}{2}\theta, \quad \text{arc } AB = r\theta.$$

But $\sin\theta = 2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta$, and therefore

$$\frac{\text{chord } AB}{\text{arc } AB} = \frac{2\sin\frac{1}{2}\theta}{\theta} = \frac{\sin\theta}{\theta} \times \frac{1}{\cos\frac{1}{2}\theta}.$$

Now, when the arc AB converges to zero, the angle θ also converges to zero and $\cos\frac{1}{2}\theta$ converges to unity. Hence, if the fraction (chord $AB \div$ arc AB) converges to unity, so does $\sin\theta/\theta$.

The discussion is made rather simpler by taking the angle AOB to be 2θ; in this case the fraction (chord $AB \div$ arc AB) is equal to $\sin\theta/\theta$.

If D is the number of degrees in the angle AOB, then

$$D = \frac{180}{\pi}\theta,\quad \frac{\sin D}{D} = \frac{\pi}{180}\cdot\frac{\sin\theta}{\theta},$$

and therefore $$\mathop{\mathrm{L}}_{D=0} \frac{\sin D}{D} = \frac{\pi}{180}. \qquad \text{(2)}$$

Again, $$\frac{\tan\theta}{\theta} = \frac{\sin\theta}{\theta} \times \frac{1}{\cos\theta},$$

and therefore $$\mathop{\mathrm{L}}_{\theta=0} \frac{\tan\theta}{\theta} = 1. \qquad \text{(3)}$$

NOTE. Unless the contrary is expressly stated, angles are measured in radians.

If the student is to do satisfactory work in differentiating and integrating circular functions he must be quite familiar with the formulae for $\sin(A \pm B)$ and $\cos(A \pm B)$. The following modifications of these formulae are important:

$$\sin A - \sin B = 2\sin\tfrac{1}{2}(A-B)\cos\tfrac{1}{2}(A+B),$$
$$\cos A - \cos B = 2\sin\tfrac{1}{2}(B-A)\sin\tfrac{1}{2}(A+B),$$
$$\sin^2 A = \tfrac{1}{2}(1-\cos 2A);\quad \cos^2 A = \tfrac{1}{2}(1+\cos 2A).$$

Example 1. Prove

$$\text{(i)} \mathop{\mathrm{L}}_{\delta x=0} \frac{2\sin(\frac{1}{2}\delta x)}{\delta x} = 1, \qquad \text{(ii)} \mathop{\mathrm{L}}_{\delta x=0} \frac{2\sin(\frac{1}{2}a\delta x)}{\delta x} = a.$$

In (i) put θ for $\frac{1}{2}\delta x$; then θ tends to 0 when δx tends to 0, and therefore

$$\mathop{\mathrm{L}}_{\delta x=0} \frac{2\sin(\frac{1}{2}\delta x)}{\delta x} = \mathop{\mathrm{L}}_{\theta=0} \frac{\sin\theta}{\theta} = 1.$$

In (ii) put θ for $\frac{1}{2}a\delta x$; then, as in case (i),

$$\mathop{\mathrm{L}}_{\delta x=0} \frac{2\sin(\frac{1}{2}a\delta x)}{\delta x} = \mathop{\mathrm{L}}_{\theta=0} a \times \frac{\sin\theta}{\theta} = a \times 1 = a.$$

Example 2. Prove

$$\text{(i)} \underset{x=0}{\mathrm{L}} \frac{\sin ax}{\sin bx}=\frac{a}{b}, \quad \text{(ii)} \underset{x=0}{\mathrm{L}} \frac{\tan ax}{\tan bx}=\frac{a}{b}.$$

We have
$$\frac{\sin ax}{\sin bx}=\frac{\sin ax}{ax}\times\frac{bx}{\sin bx}\times\frac{a}{b}.$$

When x converges to zero the first and second factors each converge to 1, while the third factor is constant; the limit is therefore a/b.

In the same way equation (ii) is established.

When x is small we may, *as an approximation*, replace $\sin ax$ or $\tan ax$ by ax; the error involved in the approximation, even if the angle be as large as 10°, is only a little greater than $\frac{1}{2}$ per cent. and 1 per cent. respectively. On the other hand, when seeking the limit for x converging to zero, of the expressions

$$\frac{\sin ax}{bx}, \quad \frac{\tan ax}{bx}, \quad \frac{\sin ax}{\tan bx}$$

we may at once, *without affecting the limit*, replace the sine and the tangent by the angle. Thus,

$$\frac{\sin ax}{bx}=\frac{\sin ax}{ax}\times\frac{ax}{bx};$$

the limit of the first factor on the right is unity, and therefore the limit of $\sin ax/bx$ is the same as that of ax/bx.

Similarly, in example 1 (ii), we may proceed thus:

$$\underset{\delta x=0}{\mathrm{L}} \frac{2\sin(\frac{1}{2}a\delta x)}{\delta x}=\underset{\delta x=0}{\mathrm{L}} \frac{2\times\frac{1}{2}a\delta x}{\delta x}=2\times\tfrac{1}{2}a=a.$$

42. Derivatives of the Circular Functions. The angle occurs so frequently in the combination $ax+b$ that the student should be familiar with the derivatives of $\sin(ax+b)$, $\cos(ax+b)$, $\tan(ax+b)$ as well as those of $\sin x$, $\cos x$, $\tan x$. Before reading the proof he should note the remarks at the end of example 2, § 41.

I. $D_x \sin x=\cos x$; $D_x \sin(ax+b)=a\cos(ax+b)$.

$$\begin{aligned} D_x \sin x &= \underset{\delta x=0}{\mathrm{L}} \frac{\sin(x+\delta x)-\sin x}{\delta x} \\ &= \underset{\delta x=0}{\mathrm{L}} \frac{2\sin(\frac{1}{2}\delta x)\cos(x+\frac{1}{2}\delta x)}{\delta x} \\ &= \underset{\delta x=0}{\mathrm{L}} \frac{2\times\frac{1}{2}\delta x\times\cos(x+\frac{1}{2}\delta x)}{\delta x} \\ &= \underset{\delta x=0}{\mathrm{L}} \cos(x+\tfrac{1}{2}\delta x) \\ &= \cos x; \end{aligned}$$

$$D_x \sin(ax+b) = \underset{\delta x=0}{\mathrm{L}} \frac{\sin[a(x+\delta x)+b]-\sin(ax+b)}{\delta x}$$
$$= \underset{\delta x=0}{\mathrm{L}} \frac{2\sin(\frac{1}{2}a\delta x)\cos(ax+\frac{1}{2}a\delta x+b)}{\delta x}$$
$$= \underset{\delta x=0}{\mathrm{L}} \frac{2\times\frac{1}{2}a\delta x\times\cos(ax+\frac{1}{2}a\delta x+b)}{\delta x}$$
$$= a\cos(ax+b).$$

The value of $D\sin(ax+b)$ may also be found by using § 26(A) Thus, let $u=ax+b$; then $y=\sin(ax+b)=\sin u$, and

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=\cos u\times a = a\cos(ax+b).$$

II. $\mathbf{D_x \cos x = -\sin x}$; $\mathbf{D_x \cos(ax+b) = -a\sin(ax+b)}$.

$$D_x \cos x = \underset{\delta x=0}{\mathrm{L}} \frac{\cos(x+\delta x)-\cos x}{\delta x} = -\underset{\delta x=0}{\mathrm{L}} \frac{\cos x-\cos(x+\delta x)}{\delta x}.$$

But $\quad \cos x - \cos(x+\delta x) = 2\sin(\frac{1}{2}\delta x)\sin(x+\frac{1}{2}\delta x)$,
and the rest of the proof is the same as for the sine. In the same way the value of $D_x\cos(ax+b)$ is obtained.

III. $\mathbf{D_x \tan x = \sec^2 x}$; $\mathbf{D_x \tan(ax+b) = a\sec^2(ax+b)}$.

$$D_x \tan x = \underset{\delta x=0}{\mathrm{L}} \frac{\tan(x+\delta x)-\tan x}{\delta x}$$
$$= \underset{\delta x=0}{\mathrm{L}} \frac{\sin(x+\delta x)\cos x-\cos(x+\delta x)\sin x}{\delta x\cos(x+\delta x)\cos x}$$
$$= \underset{\delta x=0}{\mathrm{L}} \frac{\sin\delta x}{\delta x}\times\frac{1}{\cos(x+\delta x)\cos x}$$
$$= \frac{1}{\cos^2 x} = \sec^2 x.$$

We leave it as an exercise to the student to prove that

IV. $\mathbf{D_x \operatorname{cosec} x = -\operatorname{cosec} x \cot x}$.

V. $\mathbf{D_x \sec x = \sec x \tan x}$.

VI. $\mathbf{D_x \cot x = -\operatorname{cosec}^2 x}$.

NOTE. If the angle x is not x radians but x degrees, the factor $2\sin(\frac{1}{2}\delta x)\div\delta x$ converges to $\pi/180$; therefore

$$D_x\sin x = \frac{\pi}{180}\cos x,\quad D_x\cos x = -\frac{\pi}{180}\sin x,\quad D_x\tan x = \frac{\pi}{180}\sec^2 x.$$

It is the occurrence of the factor $\pi/180$ that makes the use of degrees cumbrous in differentiation and integration.

The following exercises are very simple, but the beginner will do well to try most of them in order to fix in his memory the values of the derivatives just obtained. A careful inspection of the process and results will often give useful hints for integration.

EXERCISES XI.

Differentiate with respect to x

1. $\sin 2x$. **2.** $\cos 2x$. **3.** $\sin(2x+5)$.

4. $\cos(2x+5)$. **5.** $\sin(3-x)$. **6.** $\cos(3-x)$.

7. $\sin(\frac{1}{2}x+\frac{1}{3}\pi)$. **8.** $\cos(\frac{1}{2}x+\frac{1}{3}\pi)$. **9.** $\sin(3-2x)$.

10. $\cos(3-2x)$. **11.** $\sin 5(x-\frac{1}{2}\pi)$. **12.** $\cos 5(x-\frac{1}{2}\pi)$.

13. $\sin\frac{2\pi}{3}(x+2)$. **14.** $\cos\frac{2\pi}{3}(x+2)$. **15.** $\sin\frac{2\pi}{a}(x+b)$.

16. $\cos\frac{2\pi}{a}(x+b)$. **17.** $\sin 2x\cos x$. **18.** $\cos 2x\sin x$.

19. $\sin mx\cos nx$. **20.** $\sin mx\sin nx$. **21.** $\tan(3x-4)$.

22. $\cot(3x-4)$. **23.** $\operatorname{cosec}(2x-3)$. **24.** $\sec(3x-2)$.

25. $x\sin x$. **26.** $x\cos x$. **27.** $x^2\sin x$.

28. $x^2\cos x$. **29.** $x\sin x+\cos x$. **30.** $\sin x-x\cos x$.

31. $\frac{1}{2}x+\frac{1}{4}\sin 2x$. **32.** $\frac{1}{2}x-\frac{1}{4}\sin 2x$. **33.** $\frac{3}{4}\sin x+\frac{1}{12}\sin 3x$.

34. $\frac{1}{12}\cos 3x-\frac{3}{4}\cos x$. **35.** $\tan x-x$. **36.** $x\tan x$.

37. $\frac{\sin x}{x}$. **38.** $\frac{\cos x}{x}$. **39.** $\frac{1-\sin x}{1+\sin x}$.

40. $\frac{1-\cos x}{1+\cos x}$. **41.** $\frac{\sin x-\cos x}{\sin x+\cos x}$. **42.** $\frac{\sin\frac{1}{2}x}{1-\cos x}$.

43. $2\cos x+2x\sin x-x^2\cos x$. **44.** $\sin 2x-2x\cos 2x$.

43. Worked Examples. We shall now work one or two examples, illustrating various points.

Example 1. If $y=\sin^2(3x+4)$, find $\frac{dy}{dx}$.

Here y is a power of a function of x; denote the base of the power, namely $\sin(3x+4)$, by a single letter u, and then apply § 26 (A). Thus

$$y=u^2,\quad u=\sin(3x+4).$$

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=2u\times 3\cos(3x+4)=6\sin(3x+4)\cos(3x+4).$$

Similarly we find, for instance, the derivative of $\cos^3 x$.

$$y=u^3, \quad u=\cos x.$$

$$\frac{dy}{dx}=3u^2\times(-\sin x)=-3\cos^2 x\sin x.$$

After a little practice, the actual substitution of u will be found to be unnecessary. Thus

$$D\cos^3 x=3\cos^2 x\times(-\sin x)=-3\cos^2 x\sin x,$$

$$D\tan^2 x=2\tan x\times\sec^2 x=2\tan x\sec^2 x.$$

Example 2. Find the n^{th} derivative of $\sin(ax+b)$.

$$D\sin(ax+b)=a\cos(ax+b)=a\sin\left(ax+b+\frac{\pi}{2}\right) \quad \text{.........} (1)$$

because $\cos A=\sin(A+\tfrac{1}{2}\pi)$.

Similarly,

$$D^2\sin(ax+b)=a\,D\sin\left(ax+b+\frac{\pi}{2}\right)=a^2\sin\left(ax+b+2\frac{\pi}{2}\right),$$

$$D^3\sin(ax+b)=a^2D\sin\left(ax+b+2\frac{\pi}{2}\right)=a^3\sin\left(ax+b+3\frac{\pi}{2}\right).$$

The law of formation is now obvious; for the n^{th} derivative we find

$$D^n\sin(ax+b)=a^n\sin\left(ax+b+n\frac{\pi}{2}\right).$$

The value $a\sin(ax+b+\tfrac{1}{2}\pi)$ for the derivative of $\sin(ax+b)$ should be noted.

Example 3. If $\dfrac{dy}{dx}=\sqrt{(a^2-x^2)}$ and $x=a\sin u$, find $\dfrac{dy}{du}$ and express its value in terms of u.

$$\frac{dy}{dx}=\sqrt{(a^2-x^2)}=a\cos u, \quad \frac{dx}{du}=a\cos u;$$

therefore $$\frac{dy}{du}=\frac{dy}{dx}\times\frac{dx}{du}=a\cos u\times a\cos u=a^2\cos^2 u.$$

The value may be written

$$\frac{dy}{du}=\tfrac{1}{2}a^2(1+\cos 2u),$$

a form specially useful for integration.

Example 4. Find the turning values of $2\sin x+\sin 2x$.

Let $f(x)=2\sin x+\sin 2x$; then

$$f'(x)=2\cos x+2\cos 2x;\ f''(x)=-2\sin x-4\sin 2x.$$

Now, $$f'(x)=2(\cos x+\cos 2x)=4\cos\frac{3x}{2}\cos\frac{x}{2},$$

and therefore $f'(x)$ is zero when

$$\text{(i) } \cos\frac{3x}{2}=0 \ \text{ or (ii) } \cos\frac{x}{2}=0.$$

Restricting ourselves to values of x between 0 and 2π, we see that equation (i) gives, for $3x/2$, the values $\pi/2$, $3\pi/2$, $5\pi/2$, and therefore, for x, the values

$$\pi/3,\ \pi,\ 5\pi/3.$$

Next, $$f''\left(\frac{\pi}{3}\right) = -2\times\frac{\sqrt{3}}{2} - 4\times\frac{\sqrt{3}}{2} = -3\sqrt{3},$$

$$f''\left(\frac{5\pi}{3}\right) = -2\times\left(-\frac{\sqrt{3}}{2}\right) - 4\times\left(-\frac{\sqrt{3}}{2}\right) = +3\sqrt{3},$$

and therefore $f(\pi/3)$, which is equal to $3\sqrt{3}/2$ or $2{\cdot}598$, is a maximum and $f(5\pi/3)$, which is equal to $-2{\cdot}598$, is a minimum value of $f(x)$.

When $x=\pi$ the value of $f''(x)$ is zero, so that $f''(x)$ in this case gives no criterion. It is easy to see, however, by examining the sign of $f'(x)$ for values of x a little less and a little greater than π, that $f'(x)$ does not change sign as x increases through π. The value $f(\pi)$ is therefore not a turning value.

The values of x given by (ii) are π, 3π ... and do not give turning values of $f(x)$.

$f(x)$ is a periodic function, with period 2π, and the turning values are of course repeated in each period.

Example 5. A particle is moving in a straight line, and at time t its distance x from a fixed point O on the line is given by

$$x = a\cos(nt+\epsilon);$$

find its velocity v and its acceleration a at any instant.

We have $$x = a\cos(nt+\epsilon); \quad\text{.........(i)}$$

therefore $$v = \frac{dx}{dt} = -na\sin(nt+\epsilon) \quad\text{.........(ii)}$$

and $$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = -n^2a\cos(nt+\epsilon) \quad\text{.........(iii)}$$

or $$a = -n^2x. \quad\text{.........(iv)}$$

From (iv) we see that, at any instant, the acceleration a is proportional to the displacement x of the particle. The direction of the acceleration is towards O; because when x is positive a is negative, and when x is negative a is positive.

The motion of the particle is said to be a **simple harmonic motion** equation (iv) proves that in simple harmonic motion the acceleration is proportional to the displacement.

The acceleration is greatest (numerically) when $x=\pm a$, that is, when $nt+\epsilon$ is a multiple of π; the velocity is then zero. The velocity is greatest (numerically) when $\sin(nt+\epsilon)=\pm 1$, that is, when $nt+\epsilon$ is an odd multiple of $\pi/2$; x and a are then both zero.

Equation (iv) may be written

$$\frac{d^2x}{dt^2} = -n^2x, \text{ or } \frac{d^2x}{dt^2} + n^2x = 0. \quad\text{.........(v)}$$

This equation is called **the differential equation of simple harmonic motion**; equation (i) from which it is derived by differentiation is

called, in respect to (v), the **integral** of equation (v). The two constants a, ϵ are the constants of integration. It will be noticed that, whatever be the particular values of a and ϵ, we get the same equation (v).

The equation $x = a\sin(nt+\epsilon)$ gives rise to the same differential equation (v). (Compare Exercises XII., 56 and 57).

Example 6. A point moves so that at time t its coordinates with reference to two rectangular axes are $x = a\cos nt$, $y = b\sin nt$; find the gradient of its path at any instant.

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{nb\cos nt}{-na\sin nt} = -\frac{b}{a}\cot nt.$$

When $t=0$ we have $x=a$, $y=0$; the gradient is then infinite, so that the point is moving in a direction at right angles to the x-axis. Since dy/dt is positive, equal to nb, when $t=0$ the point is moving upwards. When $t=\pi/n$ the gradient is again infinite; but now dy/dt is negative, equal to $-nb$, and the point is moving downwards.

When $t=\pi/2n$ and $3\pi/2n$ the point is moving parallel to the x-axis, in the first case towards the left and in the second case towards the right.

By eliminating t we find the equation of the path of the point to be $x^2/a^2+y^2/b^2=1$; the path is thus an ellipse. The motion of the point is in fact compounded of two simple harmonic motions of the same period at right angles to each other.

The equation of the line through the centre of the ellipse parallel to the tangent at P (the position of the point at time t) is

$$y = -\frac{bx}{a}\cot nt.$$

This line meets the ellipse at

$$Q(-a\sin nt,\ b\cos nt) \text{ and } Q'(a\sin nt,\ -b\cos nt),$$

as may be seen by solving the equations of line and ellipse as simultaneous equations. If O is the centre of the ellipse, then

$$OQ = OQ' = \sqrt{(a^2\sin^2 nt + b^2\cos^2 nt)}.$$

The velocity of the point when at P is given by

$$\frac{ds}{dt} = \sqrt{\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}} = n\sqrt{(a^2\sin^2 nt + b^2\cos^2 nt)},$$

and is therefore proportional to OQ, the semi-diameter **conjugate** to OP.

EXERCISES. XII.

Differentiate the functions in examples 1–20.

1. $\cos^2(3x-2)$. **2.** $\sin^3(4x-1)$. **3.** $\cos^4(ax+b)$. **4.** $\sin^n(ax+b)$.

5. $\sqrt{(\sin x)}$. **6.** $\sqrt{(\cos 2x)}$. **7.** $\sin^2 x\cos^3 x$. **8.** $\sin^m x\cos^n x$.

9. $\frac{1}{\sin x}$. 10. $\frac{1}{\sin^2 x}$. 11. $\frac{\sin x}{\cos^2 x}$. 12. $\frac{\sin^m x}{\cos^n x}$.

13. $\tan^2 x$. 14. $\sec^2 x$. 15. $\tan^2(ax+b)$. 16. $x^2\tan^2 x$.

17. $x^2\sin^2 x$. 18. $x^2\cos^3 x$. 19. $x^n\sin^n x$. 20. $x^n\cos^n x$.

Find the turning points and the points of inflection on the graphs of the functions in examples 21–28; only one period of the functions need be considered. Sketch the graphs.

21. $\sin^2 x$. 22. $\sin^3 x$. 23. $\sin^4 x$. 24. $\cos^2 x$.

25. $\cos^3 x$. 26. $\cos^4 x$. 27. $\tan^2 x$. 28. $\tan^3 x$.

Find the 2nd derivative of the functions in examples 29–35.

29. $\sin^2 x$. 30. $\cos^2 x$. 31. $\tan^2 x$. 32. $\sin 5x \cos 3x$.

33. $\cos 5x \sin 3x$. 34. $\sin mx \sin nx$. 35. $\sin mx \cos nx$.

Find the n^{th} derivative of the functions in examples 36–40.

36. $\cos(ax+b)$. 37. $\sin^2 x$. 38. $\sin^3 x$.

39. $\cos^2 x$. 40. $\cos^3 x$.

41. Show that $(1+x\tan x)/x$ is a minimum when $x=\cos x$. Verify from the tables that $x=0{\cdot}739$ approximately.

42. Show that $\sin x \sin 2x$ is a maximum or a minimum when $\sin x=\sqrt{(2/3)}$, according as the angle x is acute or obtuse. State the angles to the nearest minute.

43. Show that $\sin x(1+\cos x)$ is a maximum when $x=\pi/3$.

44. Show that the maximum value of $a\sin x+b\cos x$ is $+\sqrt{(a^2+b^2)}$, and the minimum value $-\sqrt{(a^2+b^2)}$.

Solve the problem also by expressing $a\sin x+b\cos x$ in the form $k\sin(x+\theta)$.

45. Given the length (l) of an arc of a circle, show that the segment of which l is the arc will be a maximum when the segment is a semi-circle.

[If the arc subtends the angle θ at the centre of the circle, the area of the segment is

$$\tfrac{1}{2}l^2\left(\frac{1}{\theta}-\frac{\sin\theta}{\theta^2}\right).$$

This expression is a maximum when $\theta=\pi$.]

46. A circular sector has a given perimeter; show that when the area of the sector is a maximum the arc is double the radius, and that the maximum area is equal to the square on the radius.

47. From a given circular sheet of metal it is required to cut out a sector, so that the remainder can be formed into a conical vessel of maximum capacity; show that the angle of the sector removed must be $2(1-\tfrac{1}{3}\sqrt{6})\pi$ radians (about 66° 4′).

In examples 48–55 find dy/du, expressing its value in terms of u.

48. $\frac{dy}{dx}=\frac{1}{\sqrt{(a^2-x^2)}}$; $x=a\sin u$.

49. $\frac{dy}{dx}=(a^2-x^2)^{\frac{3}{2}}$; $x=a\sin u$.

50. $\frac{dy}{dx}=\frac{1}{\sqrt{(2ax-x^2)}}$; $x=a(1+\sin u)$.

51. $\frac{dy}{dx}=\sqrt{(2ax-x^2)}$; $x=a(1+\sin u)$.

52. $\frac{dy}{dx}=\frac{1}{\sqrt{\{(x-a)(b-x)\}}}$; $x=a\cos^2 u+b\sin^2 u$.

53. $\frac{dy}{dx}=\sqrt{\{(x-a)(b-x)\}}$; $x=a\cos^2 u+b\sin^2 u$.

54. $\frac{dy}{dx}=\frac{x^{\frac{5}{2}}}{(a-x)^{\frac{1}{2}}}$; $x=a\sin^2 u$.

55. $\frac{dy}{dx}=\frac{1}{(x+a)^2+b^2}$; $x+a=b\tan u$.

56. If $x=A\cos nt+B\sin nt$, show that

$$\frac{d^2x}{dt^2}+n^2x=0.$$

57. Show that each of the equations

(i) $x=A\cos(nt+B)$, (ii) $x=C\sin(nt+D)$,
(iii) $x=E\cos nt+F\sin nt$,

where A, B, C, D, E, F are constants, gives rise to the equation

$$\frac{d^2x}{dt^2}+n^2x=0.$$

58. A particle moves in a circle of radius a with constant angular velocity ω; when $t=0$, the radius to the particle makes with OX the angle ϵ. Show that, if at time t the coordinates of the particle are x, y,

$$x=a\cos(\omega t+\epsilon), \quad y=a\sin(\omega t+\epsilon).$$

Show also that

$$\frac{d^2x}{dt^2}\cos(\omega t+\epsilon)+\frac{d^2y}{dt^2}\sin(\omega t+\epsilon)=-\omega^2 a,$$

$$-\frac{d^2x}{dt^2}\sin(\omega t+\epsilon)+\frac{d^2y}{dt^2}\cos(\omega t+\epsilon)=0.$$

Prove from these equations that the acceleration of the particle is constant in magnitude and is directed to the centre of the circle.

59. The coordinates of a point are given by the equations

$$x=a(\theta-\sin\theta), \quad y=a(1-\cos\theta),$$

where $0\leqq\theta\leqq 2\pi$. Show that the tangent to the locus of the point makes with the x-axis the angle $\frac{1}{2}(\pi-\theta)$. Graph the locus. The curve is called a **cycloid**.

44. Series for $\sin x$ and $\cos x$. We shall now establish the following expressions for $\sin x$ and $\cos x$.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \pm \frac{x^{2n-1}}{(2n-1)!}; \quad \text{error} < \frac{x_1^{2n+1}}{(2n+1)!},$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \pm \frac{x^{2n}}{(2n)!}; \quad \text{error} < \frac{x_1^{2n+2}}{(2n+2)!}.$$

The signs are alternately + and −, and x_1 denotes the *numerical value* of x; that is, $x_1 = x$ when x is positive, but $x_1 = -x$ when x is negative. The symbol $n!$, where n is a positive integer, means the product of the first n natural numbers; for example,

$$3! = 1.2.3 = 6, \qquad 4! = 1.2.3.4 = 24.$$

The proof, though somewhat artificial, is very simple and depends on the following principle. If $f'(x)$ is positive and if $f(x) = 0$ when $x = 0$, then $f(x)$ is positive when $x > 0$; but if $f'(x)$ is negative and if $f(x) = 0$ when $x = 0$, then $f(x)$ is negative when $x > 0$.

The truth of the principle is established by § 19. Thus, if $f'(x)$ is positive, $f(x)$ increases as x increases; if $f(x) = 0$ when $x = 0$, then $f(x)$ becomes greater than 0, that is, $f(x)$ is positive when x becomes greater than 0. Similarly, when $f(x)$ decreases from 0 it becomes negative.

Note also that if $a - b$ is positive, a is greater than b; if $a - b$ is negative, a is less than b, the words "greater" and "less" meaning *algebraically* "greater" and "less."

Consider the following sets of functions in which x is, for the present, supposed to be positive.

$$f_1(x) = \sin x - x, \qquad F_1(x) = \cos x - \left(1 - \frac{x^2}{2}\right),$$

$$f_2(x) = \sin x - \left(x - \frac{x^3}{3!}\right), \qquad F_2(x) = \cos x - \left(1 - \frac{x^2}{2} + \frac{x^4}{4!}\right),$$

$$f_3(x) = \sin x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\right), \quad F_3(x) = \cos x - \left(1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!}\right).$$

We take the functions $f_1(x)$, $F_1(x)$, $f_2(x)$, $F_2(x)$, ... in turn and differentiate.

(i) $f_1(x)$ is *negative*, and therefore $\sin x < x$.

For, $f_1'(x) = \cos x - 1 =$ negative quantity because $\cos x < 1$. Hence $f_1(x)$ is a decreasing function: but $f_1(x) = 0$ when $x = 0$, and therefore $f_1(x)$ is negative when $x > 0$.

(ii) $F_1(x)$ is *positive*, and therefore $\cos x > 1 - \frac{x^2}{2}$.

For, $F_1'(x) = -\sin x + x =$ positive quantity because, by (i), $\sin x < x$. Hence $F_1(x)$ is an increasing function; but $F_1(x) = 0$ when $x = 0$, and therefore $F_1(x)$ is positive when $x > 0$.

(iii) $f_2(x)$ is *positive*, and therefore $\sin x > x - \frac{x^3}{3!}$.

For, $$f_2'(x) = \cos x - \left(1 - \frac{x^2}{2}\right) = F_1(x),$$ so that $f_2'(x)$ is positive, by (ii). Hence $f_2(x)$ is an increasing function; but $f_2(x) = 0$ when $x = 0$, and therefore $f_2(x)$ is positive when $x > 0$.

(iv) $F_2(x)$ is *negative*, and therefore $\cos x < 1 - \frac{x^2}{2} + \frac{x^4}{4!}$.

For, $$F_2'(x) = -\sin x + \left(x - \frac{x^3}{3!}\right) = -f_2(x),$$ so that $F_2'(x)$ is negative, by (iii). Hence $F_2(x)$ is a decreasing function; but $F_2(x) = 0$ when $x = 0$, and therefore $F_2(x)$ is negative when $x > 0$.

Proceeding in this way, we see that $f_3(x)$ is *negative* and $F_3(x)$ *positive*, so that

(v) $\sin x < x - \frac{x^3}{3!} + \frac{x^5}{5!}$; (vi) $\cos x > 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!}$.

We thus obtain the inequalities

$$x - \frac{x^3}{3!} < \sin x < x - \frac{x^3}{3!} + \frac{x^5}{5!},$$

$$1 - \frac{x^2}{2} + \frac{x^4}{4!} > \cos x > 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!}.$$

From these inequalities we get the approximations

$$\sin x = x - \frac{x^3}{3!}; \quad \text{error} < \frac{x^5}{5!},$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!}; \quad \text{error} < \frac{x^6}{6!}.$$

We may now take

$$f_4(x)=\sin x-\left(x-\frac{x^3}{3\,!}+\frac{x^5}{5\,!}-\frac{x^7}{7\,!}\right),$$

$$F_4(x)=\cos x-\left(1-\frac{x^2}{2}+\frac{x^4}{4\,!}-\frac{x^6}{6\,!}+\frac{x^8}{8\,!}\right);$$

$f_4(x)$ is positive, because $f_4'(x)=F_3(x)=$ positive quantity. The inequalities for $\sin x$ thus become

$$x-\frac{x^3}{3\,!}+\frac{x^5}{5\,!}>\sin x>x-\frac{x^3}{3\,!}+\frac{x^5}{5\,!}-\frac{x^7}{7\,!},$$

while the approximation is now

$$\sin x=x-\frac{x^3}{3\,!}+\frac{x^5}{5\,!}; \quad \text{error}<\frac{x^7}{7\,!}.$$

In the same way we find for $\cos x$

$$1-\frac{x^2}{2}+\frac{x^4}{4\,!}-\frac{x^6}{6\,!}<\cos x<1-\frac{x^2}{2}+\frac{x^4}{4\,!}-\frac{x^6}{6\,!}+\frac{x^8}{8\,!}.$$

$$\cos x=1-\frac{x^2}{2}+\frac{x^4}{4\,!}-\frac{x^6}{6\,!}; \quad \text{error}<\frac{x^8}{8\,!}.$$

The general equations for $\sin x$ and $\cos x$ may now be easily established by induction, but we shall here omit the proof.

If x is *negative*, there is no change at all as regards the cosine, because $\cos x$ is an even function of x and the powers of x are all even. As regards $\sin x$, we have merely to interchange the symbols $>$, $<$ in the inequalities. Thus, let $x=-x_1$ where x_1 is positive; then

$$x_1-\frac{x_1^{\,3}}{3\,!}<\sin x_1<x_1-\frac{x_1^{\,3}}{3\,!}+\frac{x_1^{\,5}}{5\,!},$$

that is,

$$-x+\frac{x^3}{3\,!}<-\sin x<-x+\frac{x^3}{3\,!}-\frac{x^5}{5\,!},$$

so that

$$x-\frac{x^3}{3\,!}>\sin x>x-\frac{x^3}{3\,!}+\frac{x^5}{5\,!}.$$

The approximations remain the same, x being replaced in the error term by its numerical value x_1.

It will be noticed that the approximations are alternately in excess and defect. By taking n sufficiently large the error can be made as small as we please; because, as is proved in example 1 below, the limit of $x^n/n!$ for n becoming infinite is zero, so that when n is large the error is small.

The approximations

$$\sin x = x - \tfrac{1}{6}x^3, \quad \cos x = 1 - \tfrac{1}{2}x^2$$

are often useful; even up to an angle of $\pi/6$ these approximations are very good, and amply sufficient for rough work.

Example 1. Show that $\underset{n=\infty}{\mathrm{L}} \dfrac{x^n}{n!} = 0.$

Suppose x equal to or less than the integer m; then

$$\frac{x^n}{n!} = \frac{x^m}{m!} \cdot \frac{x}{m+1} \cdot \frac{x}{m+2} \cdot \ldots \frac{x}{n} < \frac{x^m}{m!}\left(\frac{x}{m+1}\right)^{n-m}.$$

But $x/(m+1)$ is a proper fraction, and by taking n large enough we can make $\{x/(m+1)\}^{n-m}$ as small as we please.

Example 2. Calculate the sine of the radian to 7 decimal places.

In the expression for $\sin x$ let $x=1$; we shall take each term to 9 places.

$1 = 1\cdot$	$1/3! = 0\cdot166\ 666\ 667$
$1/5! = 0\cdot008\ 333\ 333$	$1/7! = 0\cdot000\ 198\ 413$
$1/9! = 0\cdot000\ 002\ 756$	$1/11! = 0\cdot000\ 000\ 025$
$1\cdot008\ 336\ 089$	$0\cdot166\ 865\ 105$

$$\sin 1 = 1\cdot008\ 336\ 089 - 0\cdot166\ 865\ 105$$
$$\doteq 0\cdot841\ 470\ 984.$$

The error is less than $1/13!$ or $1\cdot6 \times 10^{-10}$ and cannot, therefore, affect the 7^{th} decimal. Nor will the errors from neglected figures in the divisions affect the 7^{th} decimal. Hence, the sine of the radian is, to 7 decimals,

$$0\cdot8414710.$$

This example shows how rapidly the error decreases as n increases.

Example 3. **Huyghen's Rule for the Length of a Circular Arc.**

The rule is as follows: If a is the chord of the whole arc, and b the chord of half the arc, then the length (l) of the arc is $(8b-a)/3$ approximately.

Let the arc subtend at the centre of the circle an angle of θ radians, and let the radius of the circle be r; then $l=r\theta$, $a=2r\sin\frac{1}{2}\theta$, $b=2r\sin\frac{1}{4}\theta$. We now express $\sin\frac{1}{2}\theta$ and $\sin\frac{1}{4}\theta$ by means of the series for $\sin x$, putting x in turn equal to $\frac{1}{2}\theta$ and $\frac{1}{4}\theta$. Thus

$$a = 2r\{\tfrac{1}{2}\theta - \tfrac{1}{6}(\tfrac{1}{2}\theta)^3 + \tfrac{1}{120}(\tfrac{1}{2}\theta)^5 - \ldots\}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

$$b = 2r\{\tfrac{1}{4}\theta - \tfrac{1}{6}(\tfrac{1}{4}\theta)^3 + \tfrac{1}{120}(\tfrac{1}{4}\theta)^5 - \ldots\}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

Multiply (ii) by 8 and then subtract (i); we thus eliminate θ^3. Therefore

$$8b-a=2r\left\{\frac{3}{2}\theta-\frac{3\theta^5}{120\times 2^7}+\ldots\right\}$$
$$=3l\{1-\theta^4/7680+\ldots\}.$$

Hence, neglecting the fourth and higher powers of θ, we find $l=(8b-a)/3$. It may be shown that for an angle of 30° the relative error is less than 1 in 100000, for an angle of 45° less than 1 in 20000, and for an angle of 60° less than 1 in 6000.

Example 4. Prove that $\underset{x=0}{\mathrm{L}}\ \dfrac{1+\sin x-\cos x}{\sin x+\cos x-1}=1.$

If in the fraction we put $x=0$ we get 0/0, which is an undefined symbol. The fraction has however a definite limit. For, replacing $\sin x$ and $\cos x$ by the corresponding series, we see that the fraction is equal to

$$\frac{1+(x-\frac{1}{6}x^3+\ldots)-(1-\frac{1}{2}x^2+\ldots)}{(x-\frac{1}{6}x^3+\ldots)+(1-\frac{1}{2}x^2+\ldots)-1},$$

or,

$$\frac{1+\frac{1}{2}x-\frac{1}{6}x^2+\ldots}{1-\frac{1}{2}x-\frac{1}{6}x^2+\ldots},$$

and the limit of this fraction, when x converges to 0, is 1.

When an expression is not defined for a particular value of x but has a definite limit when x converges to that particular value, it is often convenient to *assign the limit as the value of the expression for that value of x.*

EXERCISES. XIII.

1. Calculate, to 7 decimal places, the cosine of the radian.

2. Show that, if powers of x above the 7^{th} are neglected,

$$\tan x=x+\frac{x^3}{3}+\frac{2x^5}{15}+\frac{17x^7}{315}.$$

[Divide the series for $\sin x$ by the series for $\cos x$.]

3. Show that, if powers of x above the 6^{th} are neglected,

$$x\cot x=1-\frac{x^2}{3}-\frac{x^4}{45}-\frac{2x^6}{945}.$$

4. Show that, if powers of x above the 6^{th} are neglected,

$$\sec x=1+\frac{x^2}{2}+\frac{5x^4}{24}+\frac{61x^6}{720}.$$

5. Show that, if powers of x above the 6^{th} are neglected,

$$x\operatorname{cosec} x=1+\frac{x^2}{6}+\frac{7x^4}{360}+\frac{31x^6}{15120}.$$

6. Find a series for $\sin^2 x$.

[Note that $\sin^2 x=\frac{1}{2}(1-\cos 2x)$.]

7. Find a series for each of the following functions:

(i) $\cos^2 x$, (ii) $\sin^3 x$, (iii) $\cos^3 x$, (iv) $\sin 3x \cos x$.

Find the limits to which the functions in examples 8-11 converge, when x converges to the value stated.

8. $\dfrac{\sin x - x}{x^3}$; $x = 0$.

9. $\dfrac{\tan x - x}{x - \sin x}$; $x = 0$.

10. $\dfrac{\tan nx - n \tan x}{n \sin x - \sin nx}$; $x = 0$.

11. $\left(\dfrac{\pi}{2} - x\right) \tan x$; $x = \dfrac{\pi}{2}$.

[In example 11 put $x = \frac{1}{2}\pi - y$; then when x tends to $\frac{1}{2}\pi$, y tends to 0.]

12. Show that if

$$A = x + \sin x - 4 \sin \tfrac{1}{2}x, \quad B = 3 + \cos x - 4 \cos \tfrac{1}{2}x,$$

then

$$A = -\tfrac{1}{12}x^3 + \text{higher powers of } x,$$

$$B = \tfrac{1}{32}x^4 + \text{higher powers of } x,$$

and find the limit of A^4/B^3 for x converging to 0.

CHAPTER X.

INTEGRATION OF DIRECT TRIGONOMETRIC FUNCTIONS. MEAN VALUES.

45. Integration of Circular Functions. From the results of differentiation (§ 42) the following integrals are deduced:

$$\int \sin x\,dx = -\cos x; \quad \int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b);$$

$$\int \cos x\,dx = \sin x; \quad \int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b);$$

$$\int \sec^2 x\,dx = \tan x; \quad \int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b);$$

$$\int \operatorname{cosec}^2 x\,dx = -\cot x; \quad \int \operatorname{cosec}^2(ax+b)\,dx = -\frac{1}{a}\cot(ax+b).$$

We again remind the student of the necessity of making himself familiar with the fundamental formulae of trigonometry; the difficulties of integration in elementary cases are more frequently due to imperfect knowledge of trigonometry than to the nature of the calculus.

Example 1. Find the area between the graph of $\sin x$, the x-axis, and that part of the graph that lies between $x=0$ and $x=\pi$.

$$\text{Area} = \int_0^\pi \sin x\,dx = \Big[-\cos x\Big]_0^\pi = (-\cos\pi) - (-\cos 0).$$

But $\cos\pi = -1$, $\cos 0 = 1$, so that the area is 2.

Example 2. Find the centroid (i) of a circular arc, (ii) of a circular sector.

Let $OA\,(=a)$ bisect the angle $BOC\,(=2\alpha)$ and take OA as the x-axis (Fig. 25). In both cases the centroid lies on OA (from symmetry).

Let $\angle XOP=\theta$, arc $AP=s$, arc $PQ=\delta s=a\delta\theta$, $x=OM=a\cos\theta$. The whole arc $CAB=2a\alpha$.

(i) For the arc CAB,

$$2a\alpha\,.\,\bar{x}=\int OM ds=\int_{-\alpha}^{\alpha}a^2\cos\theta d\theta=\Big[a^2\sin\theta\Big]_{-\alpha}^{\alpha}.$$

But $$\Big[a^2\sin\theta\Big]_{-\alpha}^{\alpha}=a^2\sin\alpha-a^2\sin(-\alpha)=2a^2\sin\alpha,$$

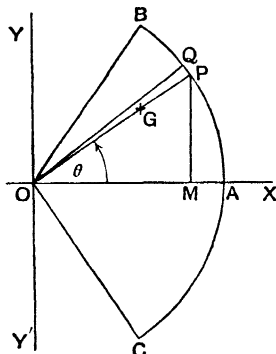


Fig. 25.

because $\sin(-\alpha)=-\sin\alpha$. The lower limit of the integral is $-\alpha$; when P lies between A and C the angle θ is negative, and θ increases from $-\alpha$ to α. Hence

$$\bar{x}=\frac{2a^2\sin\alpha}{2a\alpha}=\frac{a\sin\alpha}{\alpha}.$$

(ii) The area of the sector POQ is $\frac{1}{2}a^2\delta\theta$, and its centroid may be considered to be at G, where $OG=\frac{2}{3}OP$. The moment about OY of this elementary sector is therefore $\frac{2}{3}OM\,.\,\frac{1}{2}a^2\delta\theta$, or $\frac{1}{3}a^3\cos\theta\delta\theta$.

The area of the sector COB is $\frac{1}{2}a^2\,.\,2\alpha$, or $a^2\alpha$. Hence, for the sector,

$$a^2\alpha\,.\,\bar{x}=\int_{-\alpha}^{\alpha}\tfrac{1}{3}a^3\cos\theta d\theta=\tfrac{2}{3}a^3\sin\alpha,$$

and $$\bar{x}=\frac{2}{3}\frac{a\sin\alpha}{\alpha}.$$

Example 3. Evaluate $\int_0^{\frac{\pi}{2}}\sin^2 x dx$.

The integration of powers of $\sin x$ and $\cos x$, when the index is a small positive integer, is usually most easily effected by expressing the powers in terms of multiples of the angle. Thus,

$$\sin^2 x=\tfrac{1}{2}(1-\cos 2x),\qquad \cos^2 x=\tfrac{1}{2}(1+\cos 2x),$$
$$\sin^3 x=\tfrac{3}{4}\sin x-\tfrac{1}{4}\sin 3x,\qquad \cos^3 x=\tfrac{3}{4}\cos x+\tfrac{1}{4}\cos 3x,$$
$$\sin^4 x=\tfrac{3}{8}-\tfrac{1}{2}\cos 2x+\tfrac{1}{8}\cos 4x,\quad \cos^4 x=\tfrac{3}{8}+\tfrac{1}{2}\cos 2x+\tfrac{1}{8}\cos 4x.$$

In the present case we have

$$\int_0^{\frac{\pi}{2}}\sin^2 x dx=\int_0^{\frac{\pi}{2}}(\tfrac{1}{2}-\tfrac{1}{2}\cos 2x)dx=\Big[\tfrac{1}{2}x-\tfrac{1}{4}\sin 2x\Big]_0^{\frac{\pi}{2}}.$$

But $\sin 2x=0$ when $x=0$ and when $x=\pi/2$; the value of the integral is therefore $\pi/4$.

(See also Exercises XIV., 40–42.)

Example 4. Show from graphical considerations that

$$\text{(i)}\int_0^{\pi}\cos^3 x dx=0\,;\quad \text{(ii)}\int_0^{\pi}\cos^4 x dx=2\int_0^{\frac{\pi}{2}}\cos^4 x dx.$$

From $x=0$ to $x=\pi/2$ the ordinates of the graph of $\cos^3 x$ are positive; from $x=\pi/2$ to $x=\pi$ the ordinates are negative, but are numerically equal (in the reverse order) to the ordinates from $x=0$ to $x=\pi/2$. Hence the area represented by the integral is (algebraically) zero, so that the integral in (i) is zero.

In the graph of $\cos^4 x$, on the other hand, the ordinates are all positive (or zero), and their values in the range from $x=\pi/2$ to $x=\pi$ are equal (in the reverse order) to their values in the range from $x=0$ to $x=\pi/2$. Hence the whole area represented by the integral on the left of (ii) is twice the area represented by the integral on the right.

By considering the graphs of the integrands we can often simplify the evaluation of the integrals of trigonometric functions. (See Exercises XIV., 22-27.)

Example 5. Evaluate $\int_0^{2\pi} \sin mx \sin nx dx$, m and n being positive integers.

We have

$$\sin mx \sin nx = \tfrac{1}{2}\{\cos(m-n)x - \cos(m+n)x\}$$

when m and n are unequal; but if $m=n$, then

$$\sin mx \sin nx = \sin^2 nx = \tfrac{1}{2}(1-\cos 2nx).$$

Therefore, if m is not equal to n,

$$\int_0^{2\pi} \sin mx \sin nx dx = \left[\frac{1}{2}\left\{\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n}\right\}\right]_0^{2\pi},$$
$$=0$$

because both sines vanish when $x=0$ and when $x=2\pi$.

If $m=n$, then

$$\int_0^{2\pi} \sin mx \sin nx dx = \left[\frac{1}{2}\left(x - \frac{\sin 2nx}{2n}\right)\right]_0^{2\pi} = \frac{2\pi}{2} = \pi.$$

A similar method applies when the integrand is $\cos mx \cos nx$ or $\sin mx \cos nx$. These integrals find an important application in Fourier's Theorem (§ 52).

Example 6. Calculate $\int \sqrt{(a^2-x^2)}dx$ by means of the substitution $x=a\sin u$.

By § 43, example 3, we see that

$$\int \sqrt{(a^2-x^2)}dx = \tfrac{1}{2}a^2\int(1+\cos 2u)du = \tfrac{1}{2}a^2(u+\tfrac{1}{2}\sin 2u).$$

But
$$\tfrac{1}{2}\sin 2u = \sin u \cos u = \frac{x}{a}\cdot\frac{\sqrt{(a^2-x^2)}}{a},$$

so that
$$\int \sqrt{(a^2-x^2)}dx = \tfrac{1}{2}a^2 \sin^{-1}\frac{x}{a} + \tfrac{1}{2}x\sqrt{(a^2-x^2)}.$$

To calculate the *definite* integral $\int_0^a \surd(a^2-x^2)dx$, notice that $u=0$ when $x=0$, and $u=\pi/2$ when $x=a$; as x increases from 0 to a, u increases from 0 to $\pi/2$. Hence

$$\int_0^a \surd(a^2-x^2)dx=\tfrac{1}{2}a^2\int_0^{\frac{\pi}{2}}(1+\cos 2u)du=\frac{\pi}{4}a^2.$$

The definite integral represents quarter the area of a circle of radius a, so that the area of the circle is πa^2.

This example illustrates a substitution that is frequently effective. Compare Exercises XII., 48–55. As another illustration take the following example:

Example 7. Calculate $\displaystyle\int\frac{dx}{(x^2+4x+13)^2}$.

We have

$$x^2+4x+13=(x+2)^2+9,$$

and, by putting $3\tan u$ for $x+2$, we find

$$x^2+4x+13=9\sec^2 u.$$

Also $dx=3\sec^2 u\,du$, so that

$$\int\frac{dx}{(x^2+4x+13)^2}=\int\frac{3\sec^2 u\,du}{81\sec^4 u}=\frac{1}{27}\int\cos^2 u\,du,$$

the value of which is

$$\frac{1}{54}(u+\sin u\cos u)=\frac{1}{54}\tan^{-1}\left(\frac{x+2}{3}\right)+\frac{1}{18}\frac{x+2}{x^2+4x+13}.$$

EXERCISES. XIV.

Integrate the functions in examples 1–12.

1. $\sin 3x$.
2. $\sin(1-x)$.
3. $\cos(1-x)$.
4. $\sin\frac{2\pi}{a}(x+b)$.
5. $\cos\frac{2\pi}{a}(x+b)$.
6. $\sin^2(nx+a)$.
7. $\cos^2(nx+a)$.
8. $\tan^2 x$.
9. $\sin^2 x\cos x$.
10. $\cos^2 x\sin x$.
11. $\tan x\sec^2 x$.
12. $\cot x\,\text{cosec}^2 x$.

Calculate the definite integrals in examples 13–21.

13. $\int_0^\pi \cos^2 x\,dx$.
14. $\int_0^T \sin^2\frac{2\pi t}{T}dt$.
15. $\int_0^T \sin^2\frac{2\pi}{T}(t+\epsilon)dt$.
16. $\int_0^{\frac{\pi}{4}} \cos^2 2x\,dx$.
17. $\int_0^{\frac{2\pi}{n}} \sin^2(nt+\epsilon)dt$.
18. $\int_0^{\frac{\pi}{2}} \cos^3 x\,dx$.
19. $\int_0^{\frac{\pi}{2}} \cos^4 x\,dx$.
20. $\int_0^\pi \sin 3x\cos x\,dx$.
21. $\int_0^\pi \cos 3x\cos x\,dx$.

Prove by graphical considerations the truth of equations 22–27, n being a positive integer.

22. $\int_0^{\frac{\pi}{2}} \sin^n x\,dx = \int_0^{\frac{\pi}{2}} \cos^n x\,dx.$ 23. $\int_0^{\pi} \sin^n x\,dx = 2\int_0^{\frac{\pi}{2}} \sin^n x\,dx.$

24. $\int_0^{\pi} \cos^n x\,dx = 2\int_0^{\frac{\pi}{2}} \cos^n x\,dx$, if n is even.

25. $\int_0^{\pi} \cos^n x\,dx = 0$, if n is odd.

26. $\int_0^{2\pi} \sin^{2n} x\,dx = \int_0^{2\pi} \cos^{2n} x\,dx = 4\int_0^{\frac{\pi}{2}} \sin^{2n} x\,dx.$

27. $\int_0^{2k\pi} \sin^{2n} x\,dx = \int_0^{2k\pi} \cos^{2n} x\,dx = 4k\int_0^{\frac{\pi}{2}} \sin^{2n} x\,dx$ (k a positive integer).

28. Establish equation 22 by the substitution $x = \frac{1}{2}\pi - y$.
[$dx = -dy$; $y = \frac{1}{2}\pi$ when $x = 0$, and $y = 0$ when $x = \frac{1}{2}\pi$. Therefore

$$\int_0^{\frac{\pi}{2}} \sin^n x\,dx = -\int_{\frac{\pi}{2}}^{0} \cos^n y\,dy = \int_0^{\frac{\pi}{2}} \cos^n y\,dy = \int_0^{\frac{\pi}{2}} \cos^n x\,dx.$$

See § 31, examples 2, 5].

Evaluate the integrals in examples 29–37, employing trigonometric substitutions ; see Exercises XII. 48–55.

29. $\int_0^a (a^2 - x^2)^{\frac{3}{2}} dx.$ 30. $\int_0^a x^2 \sqrt{(a^2 - x^2)}\,dx.$ 31. $\int_0^a (a^2 - x^2)^{\frac{5}{2}} dx.$

32. $\int_0^a \frac{x^{\frac{5}{2}} dx}{(a-x)^{\frac{1}{2}}}.$ 33. $\int_0^{2a} \sqrt{(2ax - x^2)}\,dx.$ 34. $\int_0^{2a} \frac{x\,dx}{\sqrt{(2ax - x^2)}}.$

35. $\int_a^b \frac{dx}{\sqrt{\{(x-a)(b-x)\}}}.$ 36. $\int_a^b \sqrt{\{(x-a)(b-x)\}}\,dx.$ 37. $\int_1^2 \frac{dx}{(x^2 - 2x + 2)^2}.$

38. Find the surface of the prolate spheroid (Exercises IX., 17) by means of the substitution $ex = a \sin u$. Deduce the surface of a sphere of radius a.

39. Trace the curve $b^4 y^2 = x^4(a^2 - x^2)$. (The shape is that of the figure 8 laid horizontal.) Find the area of a loop of the curve.

40. Integrate the functions (i)–(iv), using the substitution $u = \sin x$:
(i) $\sin^2 x \cos^3 x$, (ii) $\sin^4 x \cos^5 x$, (iii) $\cos^3 x$, (iv) $\cos^5 x$.
[$\sin^2 x \cos^3 x = \sin^2 x \cos^2 x \cos x = \sin^2 x (1 - \sin^2 x) \cos x$,

$$\int \sin^2 x \cos^3 x\,dx = \int u^2 (1 - u^2)\,du = \tfrac{1}{3}u^3 - \tfrac{1}{5}u^5 = \tfrac{1}{3}\sin^3 x - \tfrac{1}{5}\sin^5 x.$$

This substitution is effective when the integrand is $\sin^m x \cos^n x$ and n is an *odd* positive integer ; m may have any value.]

41. Integrate the functions (i)–(iv), using the substitution $u=\cos x$:

(i) $\cos^2 x \sin^3 x$, (ii) $\cos^4 x \sin^5 x$, (iii) $\sin^3 x$, (iv) $\sin^5 x$.

[This substitution is effective when the integrand is $\sin^m x \cos^n x$ and m is an *odd* positive integer; n may have any value.]

42. Integrate the functions (i)–(iv), using the substitution $u=\tan x$:

(i) $\tan^4 x$, (ii) $\tan^6 x$, (iii) $\tan^8 x$, (iv) $\tan^{10} x$.

[$\tan^4 x = \tan^2 x . (\sec^2 x - 1) = \tan^2 x \sec^2 x - \sec^2 x + 1$,

so that $$\int \tan^4 x\, dx = \tfrac{1}{3}u^3 - u + x = \tfrac{1}{3}\tan^3 x - \tan x + x.$$

Any *even* positive power of $\tan x$ may be integrated in this way. Any even positive power of $\cot x$ (or even negative power of $\tan x$) may be integrated by using the substitution $u=\cot x$.]

46. Mean Values. The arithmetic mean of n quantities $y_1, y_2, \dots y_n$ is $(y_1+y_2+\dots+y_n)/n$. Now, let $F(x)$ be any function of x, and suppose x to vary from a to b; divide the interval $(b-a)$ into n equal parts, each equal to h, and let $y_1, y_2, \dots y_n$ be the values of $F(x)$ when x is equal to $a, a+h, \dots a+(n-1)h$ (or $b-h$) respectively. The limit for n becoming infinite (and therefore for h becoming zero) of the arithmetic mean of $y_1, y_2, \dots y_n$ is called **the mean value of the function $F(x)$ over the range $b-a$.**

This mean value may be expressed as an integral; for

$$\frac{y_1+y_2+\dots+y_n}{n} = \frac{y_1h+y_2h+\dots+y_nh}{b-a}, \quad \dots\dots\dots(1)$$

since $nh=b-a$. The numerator of the second fraction in (1) is $$F(a)h+F(a+h)h+\dots+F(b-h)h$$
and the limit of this sum for n becoming infinite is

$$\int_a^b F(x)dx. \quad \dots\dots\dots\dots\dots\dots(2)$$

Hence the mean value is

$$M=\frac{1}{b-a}\int_a^b F(x)dx. \quad \dots\dots\dots\dots\dots(3)$$

Clearly the rectangle $M(b-a)$ is equal to the area represented by the integral (2). The mean value of $F(x)$ is therefore the altitude of the rectangle whose base is the interval $(b-a)$ and whose area is equal to that included between the graph of $F(x)$, the x-axis, and the ordinates at

the ends of the interval; we might indeed take this property as the *definition* of the mean value of $F(x)$.

Example 1. The mean value of the ordinate of a semicircle of radius a is

$$\frac{1}{2a}\int_{-a}^{a}\surd(a^2-x^2)\,dx=\frac{\pi}{4}a=0\cdot7854a.$$

In this case the *diameter* is divided into n equal parts. If however the *semi-circumference* is divided into n equal parts the independent variable is the *arc* $a\theta$, measured from one end of the diameter up to the point from which the ordinate is drawn; the ordinate is $a\sin\theta$, and this mean value is, since the interval is πa,

$$\frac{1}{\pi a}\int_0^{\pi}a\sin\theta\, a d\theta=\frac{2}{\pi}a=0\cdot6366a.$$

In speaking of mean values, therefore, it is necessary to indicate clearly the independent variable—that is, the variable whose range is, in the arithmetical definition, divided into n equal parts.

Example 2. Find (i) the mean value of the ordinate, (ii) the square root of the mean value of the square of the ordinate of the curve $y=a\sin nt$ for the range from $t=0$ to $t=\pi/n$—that is, for a half period of the function $a\sin nt$.

For (i) we have, since $1\div(\pi/n)$ is equal to n/π,

$$\frac{n}{\pi}\int_0^{\frac{\pi}{n}}a\sin nt\,dt=\frac{2}{\pi}a=0\cdot6366a.$$

For (ii) the function is $y^2=a^2\sin^2 nt$; the mean value of y^2 is

$$\frac{n}{\pi}\int_0^{\frac{\pi}{n}}a^2\sin^2 nt\,dt=\frac{na^2}{2\pi}\int_0^{\frac{\pi}{n}}(1-\cos 2nt)\,dt$$
$$=\tfrac{1}{2}a^2,$$

and the square root of this mean is $a/\surd 2$—that is, $0\cdot7071a$.

This latter value is sometimes called the R.M.S. (root-mean-square) value of the ordinate; the R.M.S. value is of great importance in Alternate Current Theory.

For a *complete* period, that is, for the range from $t=0$ to $t=2\pi/n$, the mean value of $a\sin nt$ is zero, but the R.M.S. value is the same as for the half period, namely $a/\surd 2$. These results are geometrically evident.

47. Integration by Parts. In the evaluation of integrals that do not come immediately under a standard form the method of change of variable is often effective. There is another method which we shall now give, but we shall not elaborate it because, for the simple cases we here discuss, it can usually be dispensed with.

This method of *integration by parts*, as it is called, is deduced from equation (B), § 26. For the moment denote differentiation by an accent and integration by a suffix; thus

$$u' = \frac{du}{dx},\ u_1 = \int u\,dx,\ v' = \frac{dv}{dx},\ v_1 = \int v\,dx.$$

In this notation, $\frac{du_1}{dx} = u$, by the definition of an integral (§ 27).

Now, differentiate the product $u_1 v$; this gives

$$\frac{d(u_1 v)}{dx} = \frac{du_1}{dx}v + u_1\frac{dv}{dx} = uv + u_1 v'.$$

Therefore, integrating, we have

$$u_1 v = \int (uv + u_1 v')dx = \int uv\,dx + \int u_1 v'\,dx \quad \text{..........(1)}$$

or

$$\int uv\,dx = u_1 v - \int u_1 v'\,dx. \quad \text{....................(2)}$$

For a *definite* integral we have

$$\Big[u_1 v\Big]_a^b = \int_a^b (uv + u_1 v')dx = \int_a^b uv\,dx + \int_a^b u_1 v'\,dx, \quad \text{.....(3)}$$

$$\int_a^b uv\,dx = \Big[u_1 v\Big]_a^b - \int_a^b u_1 v'\,dx. \quad \text{...............(4)}$$

The rule to be obtained from (2) is complicated in expression but is easily grasped; it is applicable when the integrand is a *product*.

Example 1. Evaluate $\int x \sin x\,dx$.

Here we might take either x or $\sin x$ as the factor to be integrated (the u), but the integration of x *raises the power*, and $u_1 v'$ would be more complicated than uv. Take then $\sin x$ as u. The first step is to integrate $\sin x$ and multiply the integral of $\sin x$ by the other factor, *which is not altered till the second step.*

$$\int x \sin x\,dx = x \times (-\cos x) - \int (-\cos x) . 1 . dx$$
$$= -x\cos x + \int \cos x\,dx$$
$$= -x\cos x + \sin x.$$

Example 2. Evaluate $\int x^2 \sin x dx$.

Again take $\sin x$ for u.

$$\int x^2 \sin x dx = x^2 \times (-\cos x) - \int (-\cos x) \,.\, 2x \,.\, dx = -x^2 \cos x + 2\int x \cos x dx.$$

Now apply the rule again to $\int x \cos x dx$; we find

$$\int x \cos x dx = x \times \sin x - \int \sin x \,.\, 1 \,.\, dx = x \sin x + \cos x.$$

Therefore, substituting for $\int x \cos x dx$ the value now found, we obtain

$$\int x^2 \sin x dx = -x^2 \cos x + 2x \sin x + 2 \cos x.$$

The following results, which are deduced by the method of this article, but which would take too much space to prove, will be often of use; m and n are positive integers (see the author's *Calculus*, § 119).

$$\int_0^{\frac{\pi}{2}} \sin^n x \, dx = \int_0^{\frac{\pi}{2}} \cos^n x dx = \frac{(n-1)(n-3)\ldots}{n(n-2)(n-4)\ldots} \times \alpha, \ldots\ldots \text{(A)}$$

$$\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x \, dx$$

$$= \frac{(m-1)(m-3)\ldots \times (n-1)(n-3)\ldots}{(m+n)(m+n-2)(m+n-4)\ldots} \times \beta \ldots\ldots \text{(B)}$$

where $\alpha = 1$ when n is odd, but $\alpha = \pi/2$ when n is even; $\beta = 1$ unless m and n are *both even*, in which case $\beta = \pi/2$. Each series of factors is to be continued so long as the factors are *positive*.

Example 3. (i) $\int_0^{\frac{\pi}{2}} \sin^6 x dx = \frac{5 \,.\, 3 \,.\, 1}{6 \,.\, 4 \,.\, 2} \frac{\pi}{2} = \frac{5\pi}{32}$,

(ii) $\int_0^{\frac{\pi}{2}} \cos^7 x dx = \frac{6 \,.\, 4 \,.\, 2}{7 \,.\, 5 \,.\, 3 \,.\, 1} \times 1 = \frac{16}{35}$,

(iii) $\int_0^{\frac{\pi}{2}} \sin^4 x \cos^2 x dx = \frac{3 \,.\, 1 \times 1}{6 \,.\, 4 \,.\, 2} \frac{\pi}{2} = \frac{\pi}{32}$.

Example 4. Evaluate $\int_0^a x^4 (a^2 - x^2)^{\frac{3}{2}} dx$ by the substitution $x = a \sin u$.

$$\text{The integral} = a^8 \int_0^{\frac{\pi}{2}} \sin^4 u \cos^4 u \, du = a^8 \frac{3 \,.\, 1 \times 3 \,.\, 1}{8 \,.\, 6 \,.\, 4 \,.\, 2} \frac{\pi}{2} = \frac{3\pi a^8}{256}.$$

EXERCISES. XV.

1. A particle of mass m describes a simple harmonic motion of amplitude a and period T; show that the mean kinetic energy for the period T is half the maximum kinetic energy.

[The displacement is $x = a\cos(2\pi t/T)$, and the velocity is

$$v = \frac{dx}{dt} = -\frac{2\pi a}{T}\sin\frac{2\pi t}{T};$$

the kinetic energy is $\frac{1}{2}mv^2$, and therefore the required mean is

$$\frac{1}{T}\int_0^T \tfrac{1}{2}mv^2\,dt = \frac{m\pi^2 a^2}{T^2} = \tfrac{1}{2}\cdot\frac{2m\pi^2a^2}{T^2}.]$$

2. A particle falls freely from rest under gravity; show that the mean velocity with respect to the time of fall is half the final velocity, but that the mean velocity with respect to the distance fallen is two-thirds of the final velocity.

3. Show that in a homogeneous liquid under gravity the mean pressure-intensity over a plane area immersed in the liquid is equal to the pressure-intensity at the centroid of the area.

4. Find the R.M.S. value of i for a complete period T when

(i) $i = I\sin\left(\frac{2\pi t}{T} + \alpha\right)$,

(ii) $i = I_1\sin\left(\frac{2\pi t}{T} + \alpha_1\right) + I_2\sin\left(\frac{4\pi t}{T} + \alpha_2\right)$,

(iii) $i = I_1\sin\left(\frac{2\pi t}{T} + \alpha_1\right) + I_2\sin\left(\frac{4\pi t}{T} + \alpha_2\right) + I_3\sin\left(\frac{6\pi t}{T} + \alpha_3\right)$,

(iv) $i = \sum_{p=1}^{p=n} I_p\sin\left(\frac{2p\pi t}{T} + \alpha_p\right)$.

5. Find the mean value for a complete period T of the product $i \times e$ when

(i) $i = I\sin\frac{2\pi t}{T}$, $\quad e = E\sin\left(\frac{2\pi t}{T} + \beta\right)$,

(ii) $i = I_1\sin\left(\frac{2\pi t}{T} + \alpha_1\right) + I_2\sin\left(\frac{4\pi t}{T} + \alpha_2\right)$,

$e = E_1\sin\left(\frac{2\pi t}{T} + \beta_1\right) + E_2\sin\left(\frac{4\pi t}{T} + \beta_2\right)$.

6. If $y = A_0 + A_1\cos x + A_2\cos 2x + \ldots + A_n\cos nx$
$+ B_1\sin x + B_2\sin 2x + \ldots + B_n\sin nx$

find the mean value of (i) y, (ii) $2y\cos rx$, (iii) $2y\sin rx$, as x varies from 0 to 2π, r being any positive integer.

Evaluate the integrals 7–14 ; r is a positive integer, a is a fraction.

7. $\int_0^{\pi} x \sin rx dx.$ **8.** $\int_0^{2\pi} x \sin rx dx.$ **9.** $\int_0^{\pi} x \cos rx dx.$

10. $\int_0^{2\pi} x \cos rx dx.$ **11.** $\int_0^{\pi} x^2 \cos rx dx.$ **12.** $\int_0^{\pi} x^2 \sin rx dx.$

13. $\int_0^{\pi} \cos ax \cos rx dx.$ **14.** $\int_0^{\pi} \sin ax \sin rx dx.$

Write down the value of each of the integrals 15–24.

15. $\int_0^{\frac{\pi}{2}} \sin^8 x dx.$ **16.** $\int_0^{\frac{\pi}{2}} \sin^9 x dx.$ **17.** $\int_0^{\pi} \sin^7 x dx.$

18. $\int_0^{\pi} \cos^8 x dx.$ **19.** $\int_0^{2\pi} \sin^4 x dx.$ **20.** $\int_0^{4\pi} \cos^8 x dx.$

21. $\int_0^{\frac{\pi}{2}} \cos^3 x \sin^4 x dx.$ **22.** $\int_0^{\frac{\pi}{2}} \cos^6 x \sin^8 x dx.$

23. $\int_0^{\pi} \sin^5 x \cos^4 x dx.$ **24.** $\int_0^{2\pi} \cos^{10} x \sin^{10} x dx.$

Find the values of the integrals 25–28.

25. $\int_0^{a} x^2 \sqrt{(a^2 - x^2)} dx.$ **26.** $\int_0^{2a} x \sqrt{(2ax - x^2)} dx.$

27. $\int_0^{2a} x^2 \sqrt{(2ax - x^2)} dx.$ **28.** $\int_0^{a} x^6 \sqrt{(a^2 - x^2)} dx.$

29. Trace the curve $a^2 y^2 = x^3(2a - x)$, a being positive, and find the whole area enclosed by it.

30. Find the volume of the solid generated by the revolution of the curve $a^2 y^2 = x^3(2a - x)$ about the x-axis.

CHAPTER XI.

FOURIER SERIES.

48. Trigonometric Series. When corresponding values of two variable quantities are known it is often possible, by plotting the points which have these values as coordinates and drawing a fair curve through them, to find the equation of the curve and thus to determine the general relation that connects the two quantities. We shall now consider a different form of the same problem, namely:—given a curve, find a series of harmonic curves (that is, graphs of sines and cosines) which are such that the sum of their ordinates for any value of the abscissa will be equal to the ordinate of the given curve for that value of the abscissa. In other words, the problem is to decompose a given curve into harmonic component curves. An elementary solution of the problem for simple cases is given in the author's *Elementary Treatise on Graphs*; in the solution we shall now give, for curves whose equations are not known, the mode of presentation is due to Prof. C. Runge (*Zeitschrift für Mathematik und Physik*, vol. 48, 443–456) and to his article we refer the student for fuller information.

The analysis, though very easy, will probably be felt by the beginner to be tedious; but he should not, at the first reading, worry himself about the proof. The practical rule deduced from the analysis (§ 51) is exceedingly simple and gives with the greatest ease the required decomposition in all ordinary cases.

We require the following theorems, proved in any text-book of trigonometry.

If the angles A, $A+B$, $A+2B$, ... are in arithmetical progression with the common difference B, then

$$\sin A+\sin(A+B)+\sin(A+2B)+\dots \text{ to } n \text{ terms}$$
$$=\sin\tfrac{1}{2}nB\sin\{A+\tfrac{1}{2}(n-1)B\}\div\sin\tfrac{1}{2}B, \qquad\text{.................(1)}$$
$$\cos A+\cos(A+B)+\cos(A+2B)+\dots \text{ to } n \text{ terms}$$
$$=\sin\tfrac{1}{2}nB\cos\{A+\tfrac{1}{2}(n-1)B\}\div\sin\tfrac{1}{2}B, \qquad\text{.................(2)}$$

unless $B=2\pi$, or a multiple of 2π, when the sums are $n\sin A$ and $n\cos A$ respectively.

The following applications of (1) and (2) form the essential part of the analysis.

Let $\theta=2\pi/n$, and let r, s be positive integers *less than* $\frac{1}{2}n$.

The sum to n terms of the series

$$1+\cos r\theta\cos s\theta+\cos 2r\theta\cos 2s\theta+\cos 3r\theta\cos 3s\theta+ \qquad\text{............(3)}$$

is zero if s is not equal to r, but is $\frac{1}{2}n$ if $s=r$.

Write (3) in the form

$$\tfrac{1}{2}\{1+\cos(s-r)\theta+\cos 2(s-r)\theta+\cos 3(s-r)\theta+\dots \text{ to } n \text{ terms}\}$$
$$+\tfrac{1}{2}\{1+\cos(s+r)\theta+\cos 2(s+r)\theta+\cos 3(s+r)\theta+\dots \text{ to } n \text{ terms}\}.$$

If s is not equal to r we may apply (2) to each series. For the first series let $A=0$, $B=(s-r)\theta$,

$$\text{then } \sin\tfrac{1}{2}nB=\sin\tfrac{1}{2}n(s-r)\theta=\sin(s-r)\pi=0,$$

while $\sin\frac{1}{2}B$ is not zero. Thus the first series is zero, and similarly the second is seen to be zero.

If $s=r$, the second series is zero, but each term in the first brackets is 1, so that the sum (3) is $\frac{1}{2}n$.

We require also that case of (3) for which $r=s=\frac{1}{2}n$. In this case (3) takes the form

$$1+\cos^2(\tfrac{1}{2}n\theta)+\cos^2(2\,.\,\tfrac{1}{2}n\theta)+\cos^2(3\,.\,\tfrac{1}{2}n\theta)+\dots \text{ to } n \text{ terms} \qquad\text{......(3)}'$$
$$=1+\cos^2\pi+\cos^2 2\pi+\cos^2 3\pi+\dots \text{ to } n \text{ terms}$$
$$=n.$$

In a similar way it may be shown that when $\theta=2\pi/n$ and r, s are positive integers less than $\frac{1}{2}n$,

$$\sin r\theta\sin s\theta+\sin 2r\theta\sin 2s\theta+\sin 3r\theta\sin 3s\theta+\dots \text{ to } (n-1) \text{ terms} \qquad\text{...(4)}$$

is zero, if s is not equal to r, but is $\frac{1}{2}n$ if $s=r$;

$$\sin r\theta\cos s\theta+\sin 2r\theta\cos 2s\theta+\sin 3r\theta\cos 3s\theta+\dots \text{ to } (n-1) \text{ terms} \qquad\text{...(5)}$$

is zero, both if s is not equal to r and if $s=r$.

To bring (4) and (5) under (1) and (2), let $A=0$.

Let us now suppose a curve to be given, the abscissae ranging from $x=0$ to $x=2\pi$. The problem is, to determine the values of the coefficients A_0, A_1, A_2, ... B_1, B_2, ... in the equation

$$y=A_0+A_1\cos x+A_2\cos 2x+\dots+A_n\cos nx+\dots$$
$$+B_1\sin x+B_2\sin 2x+\dots+B_n\sin nx+\dots, \qquad\text{...(F)}$$

so that, for every value of x between **0** and $\mathbf{2\pi}$, the value of y calculated from the equation shall be equal to the corresponding value of y given by the curve.

The terms $A_1\cos x$ and $B_1\sin x$ are called the first, or fundamental, harmonics; the terms $A_2\cos 2x$ and $B_2\sin 2x$ the second harmonics; and, in general, the terms $A_n\cos nx$ and $B_n\sin nx$ the n^{th} harmonics. When n is an odd integer the harmonics are said to be odd; when n is an even integer the harmonics are said to be even.

We can always find C_n and D_n, or C'_n and D'_n, so that, for every n,

$$A_n\cos nx + B_n\sin nx = C_n\cos(nx + D_n),$$

or,

$$A_n\cos nx + B_n\sin nx = C'_n\sin(nx + D'_n);$$

but it is more convenient for the calculations to keep the terms as in (F).

In general, the series in (F) is an infinite series, and where the equation of the curve is given in the form $y=f(x)$ the coefficients can be determined by integration (§ 52). In many practical cases, however, the equation of the curve is not known. In such cases a limited number of abscissae, $x_1, x_2, \ldots$ is chosen, the ordinates $y_1, y_2, \ldots$ at $x_1, x_2, \ldots$ are read off the graph, and the values x_1, y_1, then $x_2, y_2, \ldots$ inserted in (F). We thus obtain as many equations as we please; n equations will enable us to calculate n of the coefficients $A_0, A_1, A_2, \ldots B_1, B_2, \ldots$ and the n terms so obtained furnish an approximate solution.

49. Twelve Equidistant Ordinates. When the number of the equations just referred to is large, the calculations are laborious; but by a proper choice of the abscissae $x_1, x_2, \ldots$ the work can be simplified. A solution that is very simple in practice, and that is also fairly general, is obtained by choosing 12 *equidistant* ordinates. In this case the interval from 0 to 2π is divided into 12 equal parts, each part being $2\pi/12$ or $\pi/6$. Denote $\pi/6$ by θ; then the 12 values chosen for x are

$$0,\ \theta,\ 2\theta,\ 3\theta, \ldots 11\theta;$$

the values of y corresponding to these values of x may be denoted by

$$y_0,\ y_1,\ y_2,\ y_3, \cdots y_{11}$$

respectively.

From these 12 pairs of values we obtain 12 equations and therefore 12 coefficients. We take as the approximate value of y,

$$y = A_0 + A_1\cos x + A_2\cos 2x + A_3\cos 3x$$
$$+ A_4\cos 4x + A_5\cos 5x + A_6\cos 6x$$
$$+ B_1\sin x + B_2\sin 2x + B_3\sin 3x + B_4\sin 4x + B_5\sin 5x. \quad (1)$$

Equation (1) may be written more compactly thus,

$$y = A_0 + \sum_{r=1}^{r=6} A_r\cos rx + \sum_{r=1}^{r=5} B_r\sin rx. \quad \text{............}(1')$$

Now let $s\theta$ be any one of the 12 values of x, and y_s the corresponding value of y; inserting $s\theta$ for x, and y_s for y in (1′), we get

$$y_s = A_0 + \sum_{r=1}^{r=6} A_r\cos rs\theta + \sum_{r=1}^{r=5} B_r\sin rs\theta. \quad \text{...........}(2)$$

Equation (2) represents 12 equations, obtained by giving to s in succession the values 0, 1, 2, ... 11; the equations so obtained may be called respectively the first, second, third,... twelfth of equations (2). The beginner should write out several of these in full; for example, the second equation of (2) is

$$y_1 = A_0 + A_1\cos\theta + A_2\cos 2\theta + A_3\cos 3\theta$$
$$+ A_4\cos 4\theta + A_5\cos 5\theta + A_6\cos 6\theta$$
$$+ B_1\sin\theta + B_2\sin 2\theta + B_3\sin 3\theta + B_4\sin 4\theta + B_5\sin 5\theta.$$

The principle of the solution is simply this. To obtain the value of any one of the letters, say A_4, multiply *each* of equations (2) by the coefficient of A_4 *in that equation* and then add the 12 equations thus obtained, putting into brackets the twelve coefficients of each of the letters $A_0, A_1, A_2 \ldots$. It will be found, by § 48, equations (1)-(5), that every coefficient *except that of the selected letter* will vanish, while the coefficient of the selected letter is, by § 48 (3), (3′) or (4), either $\frac{1}{2}n$ or n, that is, in this case 6 or 12; when the selected letter is A_4, the coefficient is 6.

To obtain A_0, the coefficient of which is simply unity, add all the equations (2); we find

$$12A_0 = y_0 + y_1 + y_2 + \ldots + y_{11} = \sum_{s=0}^{s=11} y_s. \quad \text{...........}(3)$$

In this case the coefficient of every letter other than A_6 vanishes by § 48 (1) or (2).

To obtain A_6, multiply the first of equations (2) by 1, the second by $\cos 6\theta$, the third by $\cos 12\theta$ ... and then add; we find (§ 48 (3′)),

$$12A_6 = y_0 - y_1 + y_2 - y_3 + \dots + y_{10} - y_{11},$$

which may be written

$$12A_6 = \sum_{s=0}^{s=11} y_s \cos s\pi. \quad \dots\dots\dots\dots\dots\dots\dots (4)$$

Notice that this is the case of § 48 (3′); $6 = \frac{1}{2}n$, n being here 12. Also $6\theta = \pi$, so that $\cos 6\theta = -1$, $\cos 12\theta = 1$, ... The coefficient of every B vanishes by § 48 (5).

To obtain A_r, where $r = 1, 2, 3, 4, 5$, multiply the first of equations (2) by 1, the second by $\cos r\theta$, the third by $\cos 2r\theta$, ... and then add; we find

$$6A_r = y_0 + y_1 \cos r\theta + y_2 \cos 2r\theta + \dots = \sum_{s=0}^{s=11} y_s \cos rs\theta. \dots (5)$$

In this case the coefficient of every A, *except the one selected*, vanishes by § 48 (3), and the coefficient of every B vanishes by § 48 (5).

To obtain B_r, where $r = 1, 2, 3, 4, 5$, multiply the first of equations (2) by 0 (the coefficient of every B in the first of (2) is zero), the second by $\sin r\theta$, the third by $\sin 2r\theta$, ... and then add; we find

$$6B_r = y_1 \sin r\theta + y_2 \sin 2r\theta + \dots = \sum_{s=1}^{s=11} y_s \sin rs\theta. \dots\dots (6)$$

50. Simplifications. The values found in equations (3)-(6) of § 49 can be greatly simplified; it should first be noted that A_0 is the arithmetic mean, or the average, of the 12 ordinates.

We can group together y_1 and y_{11}, y_2 and y_{10}, y_3 and y_9, y_4 and y_8, y_5 and y_7; in each pair the sum of the suffixes is 12. For, remembering that $\theta = \pi/6$, we have

$$11r\theta = 12r\theta - r\theta = 2r\pi - r\theta; \qquad 10r\theta = 2r\pi - 2r\theta;$$

$$9r\theta = 2r\pi - 3r\theta; \quad 8r\theta = 2r\pi - 4r\theta; \quad 7r\theta = 2r\pi - 5r\theta.$$

Therefore in (5) the coefficients of y_1 and y_{11}, y_2 and y_{10}, ... are equal, while in (6) these coefficients are numerically equal but of opposite sign.

The terms y_0, $y_6\cos 6r\theta$ do not come under the grouping; the term $y_6\sin 6r\theta$ is zero, so that y_6 disappears from the value of B_r.

Take now the notations

$y_1+y_{11}=a_1$, $y_2+y_{10}=a_2$, $y_3+y_9=a_3$, $y_4+y_8=a_4$, $y_5+y_7=a_5$,
$y_1-y_{11}=b_1$, $y_2-y_{10}=b_2$, $y_3-y_9=b_3$, $y_4-y_8=b_4$, $y_5-y_7=b_5$,
and also for symmetry, $y_0=a_0$, $y_6=a_6$.

We now obtain if $r=1, 2, 3, 4, 5$,

$$6A_r=\sum_{s=0}^{s=6}a_s\cos rs\theta \ \ldots(7); \qquad 6B_r=\sum_{s=1}^{s=5}b_s\sin rs\theta\ldots.(8);$$

while, $$12A_0=\sum_{s=0}^{s=6}a_s\ldots\ldots.(9); \qquad 12A_6=\sum_{s=0}^{s=6}a_s\cos s\pi\ldots(10)$$

But the terms can be still further grouped. When r is *odd*, $r=1, 3, 5$, we have

$$\cos 6r\theta=-1;\ \cos 5r\theta=-\cos r\theta;$$
$$\cos 4r\theta=-\cos 2r\theta;\ \cos 3r\theta=0;$$
$$\sin 5r\theta=\sin r\theta;\ \sin 4r\theta=\sin 2r\theta;\ \sin 3r\theta=\sin 3r\theta.$$

When r is *even*, $r=2, 4$, we have

$$\cos 6r\theta=1;\ \cos 5r\theta=\cos r\theta;$$
$$\cos 4r\theta=\cos 2r\theta;\ \cos 3r\theta=\cos 3r\theta;$$
$$\sin 5r\theta=-\sin r\theta;\ \sin 4r\theta=-\sin 2r\theta;\ \sin 3r\theta=0.$$

Take now the notations

$a_0+a_6=c_0$, $a_1+a_5=c_1$, $a_2+a_4=c_2$, $a_3=c_3$,
$a_0-a_6=c_0'$, $a_1-a_5=c_1'$, $a_2-a_4=c_2'$.
$b_1+b_5=d_1$, $b_2+b_4=d_2$, $b_3=d_3$,
$b_1-b_5=d_1'$, $b_2-b_4=d_2'$.

The formulae become

r *odd*, $r=1, 3, 5$; $6A_r=c_0'+c_1'\cos r\theta+c_2'\cos 2r\theta$, ...(11)
$6B_r=d_1\sin r\theta+d_2\sin 2r\theta+d_3\sin 3r\theta$...(12)

r *even*, $r=2, 4$; $6A_r=c_0+c_1\cos r\theta+c_2\cos 2r\theta+c_3\cos 3r\theta$,(11′)
$6B_r=d_1'\sin r\theta+d_2'\sin 2r\theta$; ....(12′)

while for $r=0, 6$, we have

$$12A_0=c_0+c_1+c_2+c_3\ \ldots(13); \quad 12A_6=c_0-c_1+c_2-c_3.\ldots(14)$$

The only acute angles required for the evaluation are $\pi/6$ and $\pi/3$. By giving to r its various values we find that the coefficients are given by the equations

$$12A_0=(c_0+c_2)+(c_1+c_3);\quad 12A_6=(c_0+c_2)-(c_1+c_3);$$

$$6A_1=(c_0'+\tfrac{1}{2}c_2')+\frac{\sqrt{3}}{2}c_1';\quad 6A_5=(c_0'+\tfrac{1}{2}c_2')-\frac{\sqrt{3}}{2}c_1';$$

$$6A_2=(c_0-\tfrac{1}{2}c_2)+(\tfrac{1}{2}c_1-c_3);\quad 6A_4=(c_0-\tfrac{1}{2}c_2)-(\tfrac{1}{2}c_1-c_3);$$

$$6A_3=c_0'-c_2'.$$

$$6B_1=(\tfrac{1}{2}d_1+d_3)+\frac{\sqrt{3}}{2}d_2;\quad 6B_5=(\tfrac{1}{2}d_1+d_3)-\frac{\sqrt{3}}{2}d_2;$$

$$6B_2=\frac{\sqrt{3}}{2}(d_1'+d_2');\quad 6B_4=\frac{\sqrt{3}}{2}(d_1'-d_2');$$

$$6B_3=d_1-d_3.$$

When thus reduced, the calculations are extremely simple; the only calculation that is not easy to effect mentally is the multiplication by $\sqrt{3}/2$. The terms $6A_1$ and $6A_5$, $6A_2$ and $6A_4$, ... are obtained respectively as the sum and the difference of the same terms. In the next article we give Runge's scheme (slightly altered), which enables the solution to be effected with great ease in any particular case.

51. Runge's Scheme for Solution. First write the ordinates so that $y_0, y_1, \ldots y_6$ are in a row and $y_7, y_8, \ldots y_{11}$ in another row, placing the pairs y_1 and y_{11}, y_2 and y_{10}, y_3 and y_9 ... in the same column. Then add and subtract; when there is only one term in a column it appears only in the sum. The arrangement is then as follows:

	y_0,	y_1,	y_2,	y_3,	y_4,	y_5,	y_6
		y_{11},	y_{10},	y_9,	y_8,	y_7	
Sum	a_0,	a_1,	a_2,	a_3,	a_4,	a_5,	a_6
Difference		b_1,	b_2,	b_3,	b_4,	b_5	

Here $a_0=y_0$, $a_1=y_1+y_{11}$, $b_1=y_1-y_{11}$, ... In writing the rows, we continue the first row till y_6 is reached; then we write $y_7, y_8, \ldots$ from right to left, placing y_7 under y_5, and continue till y_{11} is reached.

We next find the c's and d's in the same way.

	$a_0,\ a_1,\ a_2,\ a_3$	$b_1,\ b_2,\ b_3$
	$a_6,\ a_5,\ a_4$	$b_5,\ b_4$
Sum	$c_0,\ c_1,\ c_2,\ c_3$	Sum $d_1,\ d_2,\ d_3$
Diff.	$c_0',\ c_1',\ c_2'$	Diff. $d_1',\ d_2'$

Having calculated the c's and d's, we arrange the equations for finding the coefficients A_0, A_1, ... of the cosine terms as follows:

COSINE TERMS.

r	0 and 6		1 and 5		2 and 4		3
	c_0 c_2	c_1 c_3	c_0' $\frac{1}{2}c_2'$	$\frac{\sqrt{3}}{2}c_1'$	c_0 $-\frac{1}{2}c_2$	$\frac{1}{2}c_1$ $-c_3$	$c_0'-c_2'$
1st Column - 2nd Column -							
Sum of Cols. - Diff. of Cols. -	$12A_0$ $12A_6$		$6A_1$ $6A_5$		$6A_2$ $6A_4$		$6A_3$

Thus, to find A_0 and A_6 take the division headed 0, 6; add the terms in the first column, getting c_0+c_2; add those in the second, getting c_1+c_3; then the sum of these two results, namely $(c_0+c_2)+(c_1+c_3)$, is the value of $12A_0$, while the difference, namely $(c_0+c_2)-(c_1+c_3)$, is the value of $12A_6$. In the same way the other coefficients are found.

There is a similar form for the coefficients B_1, B_2 ... of the sine terms.

SINE TERMS.

r	1 and 5		2 and 4		3
	$\frac{1}{2}d_1$ d_3	$\frac{\sqrt{3}}{2}d_2$	$\frac{\sqrt{3}}{2}d_1'$	$\frac{\sqrt{3}}{2}d_2'$	d_1-d_3
1st Column - 2nd Column -					
Sum of Cols. - Diff. of Cols. -	$6B_1$ $6B_5$		$6B_2$ $6B_4$		$6B_3$

The following example will show the simplicity of the solution.

Example. Let the ordinates be 13, 18, 20, 18, 10, -1, -10, -14, -15, -14, -10, 3.

	13,	18,	20,	18,	10,	-1,	-10	
		3,	-10,	-14,	-15,	-14		
Sum	13,	21,	10,	4,	-5,	-15,	-10	(a)
Diff.		15,	30,	32,	25,	13		(b)

	13,	21,	10, 4				15,	30,	32	
	-10,	-15,	-5				13,	25		
Sum	3,	6,	5, 4	(c)		Sum	28,	55,	32	(d)
Diff.	23,	36,	15	(c')		Diff.	2,	5		(d')

The letters (a), (b), (c) ... at the end of the rows suggest that the numbers in these rows belong respectively to those denoted in the scheme by a's, b's, c's, ...

COSINE TERMS.

r	0 and 6.		1 and 5.		2 and 4.		3.
	3 5	6 4	23 7·5	31·2	3 $-2·5$	3 -4	$23-15$
1st Col. 2nd Col.	8 10		30·5 31·2		0·5 -1		
Sum - Diff. -	$18=12A_0$ $-2=12A_6$		$61·7=6A_1$ $-0·7=6A_5$		$-0·5=6A_2$ $1·5=6A_4$		$8=6A_3$

SINE TERMS.

r	1 and 5.		2 and 4.		3
	14 32	47·6	1·7	4·3	$28-32$
1st Col. 2nd Col.	46 47·6		1·7 4·3		
Sum - Diff. -	$93·6=6B_1$ $-1·6=6B_5$		$6·0=6B_2$ $-2·6=6B_4$		$-4=6B_3$

Hence

$A_0=1{\cdot}5,\ A_1=10{\cdot}3,\ A_2=-0{\cdot}1,\ A_3=1{\cdot}3,\ A_4=0{\cdot}2,\ A_5=-0{\cdot}1,\ A_6=-0{\cdot}2,$
$B_1=15{\cdot}6,\ B_2=1{\cdot}0,\ B_3=-0{\cdot}7,\ B_4=-0{\cdot}4,\ B_5=-0{\cdot}3$

where the values are given to only one decimal place.

The method of verification stated in Exercises XVI., 5, is worthy of notice.

52. Fourier's Series. The solution stated in § 51 will be found sufficient for most of the cases in which a curve has to be analysed into harmonic components by means of a limited number of selected ordinates. It is not hard to work out formulae when instead of 12 we have, say, 36 ordinates, though the arithmetic is certainly tedious. (See Exercises XVI., 6.)

We shall now suppose that the equation of the curve is given in the form $y=f(x)$. Let the interval from 0 to 2π be divided into $(2n+1)$ equal parts, each part being $2\pi/(2n+1)$. Denote $2\pi/(2n+1)$ by θ; then the $(2n+1)$ values of x will be

$$0,\ \theta,\ 2\theta,\ 3\theta, \dots 2n\theta;$$

and the values of y corresponding to these values of x are

$$f(0),\ f(\theta),\ f(2\theta),\ f(3\theta), \dots f(2n\theta),$$

or, for brevity, $\quad y_0,\ y_1,\ y_2,\ y_3, \dots y_{2n}$.

Assume as equation for y

$$y=A_0+\sum_{r=1}^{r=n} A_r \cos rx+\sum_{r=1}^{r=n} B_r \sin rx, \quad \dots\dots\dots\dots(1)$$

in which there are $(2n+1)$ coefficients $A_0, A_1, A_2, \dots B_1, B_2, \dots$.

By exactly the same analysis as in § 49, we find

$$\frac{2n+1}{2}A_r=\sum_{s=0}^{s=2n} y_s \cos rs\theta; \dots(2) \quad \frac{2n+1}{2}B_r=\sum_{s=1}^{s=2n} y_s \sin rs\theta; \dots(3)$$

$$(2n+1)A_0=\sum_{s=0}^{s=2n} y_s. \quad \dots\dots\dots\dots\dots\dots(4)$$

We shall now find the limit of these expressions for n becoming infinite.

Multiply each side of (2) by $2\pi/(2n+1)$ or θ, and write $f(s\theta)$ for y_s; we thus obtain the equation

$$\pi A_r=\sum_{s=0}^{s=2n} f(s\theta)\cos rs\theta \,.\, \theta. \quad \dots\dots\dots\dots\dots\dots(2')$$

Now, let $s\theta = u$, $(s+1)\theta = u + \delta u$, so that $\theta = \delta u$; when $s=0$, $u=0$, and when $s=2n$, $u=2n\theta=2\pi-\theta$. Equation (2′) now becomes

$$\pi A_r = \sum_{u=0}^{u=2\pi-\theta} f(u)\cos ru\,\delta u. \quad \text{.................}(2'')$$

But when n becomes infinite, θ becomes 0 and the sum in (2″) becomes a definite integral with respect to u, from $u=0$ to $u=2\pi$. Therefore

$$\pi A_r = \int_0^{2\pi} f(u)\cos ru\,du \text{..................}(5)$$

In exactly the same way we find

$$\pi B_r = \int_0^{2\pi} f(u)\sin ru\,du \ldots(6);\quad 2\pi A_0 = \int_0^{2\pi} f(u)\,du.\ \ldots(7)$$

The sum on the right hand side of (1) is now an infinite series, called **Fourier's series**, and the values of the coefficients are given by the integrals (5), (6), (7).

We may however obtain the coefficients without going through the analysis just given. In the first place observe that

$$\int_0^{2\pi}\cos rx\cos sx\,dx = 0, \text{ if } s \neq r, \quad = \pi, \text{ if } s = r; \qquad \int_0^{2\pi}\sin rx\sin sx\,dx = 0, \text{ if } s \neq r, \quad = \pi, \text{ if } s = r;$$

$$\int_0^{2\pi}\cos rx\sin sx\,dx = 0;\quad \int_0^{2\pi}\cos rx\,dx = 0 = \int_0^{2\pi}\sin rx\,dx.$$

These results are proved, as in § 45, example 5.

Now to find A_r, multiply equation (1) by $\cos rx$, the coefficient of A_r, and integrate both sides from 0 to 2π. The integral of every term on the right, except that which contains A_r, will vanish because each term gives an integral of the type just written down. The term that contains A_r is $A_r\cos^2 rx$, and the integral of this is πA_r.

Hence
$$\int_0^{2\pi} f(x)\cos rx\,dx = \pi A_r,$$

which is the same equation as (5). Of course, the definite integral is the same whether the variable of integration is x or u.

In the same way we find B_r, by multiplying both sides of (1) by $\sin rx$, and integrating from 0 to 2π; we find A_0 by simply integrating each side of (1) from 0 to 2π.

Example 1. Find the values of A_0, A_1, A_2, ... B_1, B_2, ... if

$$x = A_0 + A_1\cos x + A_2\cos 2x + \ldots + B_1\sin x + B_2\sin 2x + \ldots \ldots\ldots\text{(i)}$$

where the cosine series and the sine series are each continued to infinity.

In this case $f(x) = x$. To obtain A_0, integrate each side of (i) from 0 to 2π; therefore

$$\int_0^{2\pi} x\,dx = \int_0^{2\pi} A_0\,dx, \text{ or } A_0 = \pi.$$

The integral of $A_1\cos x$, $A_2\cos 2x$, ... is zero, so that only the integral of A_0 is left.

To obtain A_r, where r has any one of the values 1, 2, 3, ... , multiply each side of (i) by $\cos rx$ and integrate from 0 to 2π. (If $r=1$, we multiply by $\cos x$; if $r=10$, we multiply by $\cos 10x$ and so on.) Now

$$\int_0^{2\pi} x\cos rx\,dx = \left[x \times \frac{\sin rx}{r} + \frac{\cos rx}{r^2}\right]_0^{2\pi} = 0.$$

The integral of the right side of (i) after multiplication by $\cos rx$ is πA_r; therefore $0 = \pi A_r$ or $A_r = 0$.

Hence no cosine term occurs in the series.

To obtain B_r, multiply each side of (i) by $\sin rx$ and integrate from 0 to 2π; we find

$$\pi B_r = \int_0^{2\pi} x\sin rx\,dx = \left[x \times \frac{-\cos rx}{r} + \frac{\sin rx}{r^2}\right]_0^{2\pi} = -\frac{2\pi}{r},$$

and therefore $$B_r = -\frac{2}{r}.$$

Equation (i) now becomes

$$x = \pi - \frac{2}{1}\sin x - \frac{2}{2}\sin 2x - \frac{2}{3}\sin 3x - \ldots - \frac{2}{n}\sin nx - \ldots, \ldots\ldots\text{(ii)}$$

or, $$x = \pi - 2\left(\frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \ldots + \frac{\sin nx}{n} + \ldots\right).$$

It is interesting to compare this solution with that obtained by the method of § 51. The 12 ordinates are

$$0,\ \frac{\pi}{6},\ \frac{2\pi}{6},\ \frac{3\pi}{6},\ \ldots\ \frac{11\pi}{6}.$$

and it will be found that

$$A_0 = \frac{11\pi}{12},\ A_1 = A_2 = A_3 = A_4 = A_5 = -\frac{\pi}{6},\ A_6 = -\frac{\pi}{12},$$

$$B_1 = -\frac{2+\sqrt{3}}{6}\pi = -1{\cdot}95,\ B_2 = -\frac{\sqrt{3}}{6}\pi = -0{\cdot}91,\ B_3 = -\frac{\pi}{6} = -0{\cdot}52,$$

$$B_4 = -\frac{\sqrt{3}}{18}\pi = -0{\cdot}30,\ B_5 = -\frac{2-\sqrt{3}}{6}\pi = -0{\cdot}14.$$

These values for A_0, B_1, B_2, B_3, B_4, B_5 are fair approximations to the values given in (ii), but it will be noticed that cosine terms appear with five coefficients equal to $-0{\cdot}52$ and one equal to $-0{\cdot}26$.

Example 2. Find the value of the series

$$1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\tfrac{1}{9}-\tfrac{1}{11}+\ldots \text{ to infinity.}$$

Write equation (ii) of example 1 in the form

$$\frac{\pi-x}{2}=\sin x+\tfrac{1}{2}\sin 2x+\tfrac{1}{3}\sin 3x+\tfrac{1}{4}\sin 4x+\ldots.$$

Now let $x=\pi/2$; when the angle is an even multiple of x the sine will be zero, but when it is an odd multiple of x the sine will be $+1$ or -1. The sine will be $+1$ when the multiple is 1, 5, 9, ... but -1 when the multiple is 3, 7, 11, ...

Also when $x=\pi/2$ we have $(\pi-x)/2=\pi/4$. Therefore

$$\frac{\pi}{4}=1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\tfrac{1}{9}-\tfrac{1}{11}+\ldots.$$

53. Validity of the Fourier Series. The discussion of § 52 suggests that, for values of x from $x=0$ to $x=2\pi$, a function $f(x)$ can be represented by an infinite series of the form

$$f(x)=A_0+\sum_{r=1}^{r=\infty}A_r\cos rx+\sum_{r=1}^{r=\infty}B_r\sin rx \quad \ldots\ldots\ldots\ldots(1)$$

where A_0, A_r, B_r have the values given in § 52. A rigorous proof of the suggestion is, however, too difficult to be given here; but we shall state some restrictions that must be remembered in applying the theorem.

(i) The function $f(x)$ is to be single-valued, finite and, *in general*, continuous. In other words, the graph of $f(x)$ must have only one ordinate for each value of the abscissa (a parallel to the y-axis must cut the curve in only one point); no ordinate must be infinite; and there must be no break in the curve (§ 10).

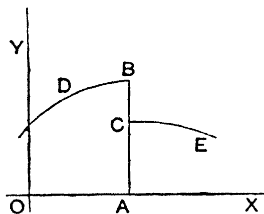


Fig. 26.

We say that the function must be *in general* continuous; but certain kinds of breaches of continuity may occur, of which the most important is that shown in Fig. 26. Here, the ordinate at $x=OA=a$ is equal to AB when x approaches a from the left, but is equal to AC when x approaches a from the right. The value of the series on

the right of (1), when $x=a$, is in this case *neither AB nor AC, but is half the sum of AB and AC.*

(ii) Equation (1) is only true, in general, for values of x between 0 and 2π. The series in (1) is a **periodic function** of x, the period being 2π; that is, the value of the series is the same for x equal to $x_1 \pm 2\pi$, $x_1 \pm 4\pi, \ldots$ as for $x=x_1$. The complete graph of the series consists of the graph of the portion from $x=0$ to $x=2\pi$ and its repetitions infinitely often, to the right from 2π to 4π, from 4π to 6π, ... and to the left from 0 to -2π, from -2π to -4π, ... Hence, unless $f(x)$ is itself a periodic function with period 2π, the function on the left of equation (1) will not be equal to the series on the right *except for values of x between 0 and 2π.*

For instance, the function x in § 52, example 1, is represented by a straight line of unlimited length AOB (Fig. 27). The series on the right of equation (ii), on the other hand, is represented by the portion OC and its repetitions DE, HK, etc.

Fig. 27.

(iii) If $f(0)$ and $f(2\pi)$ are not equal, the series in (1) is *not equal to $f(0)$ when $x=0$, nor to $f(2\pi)$ when $x=2\pi$, but is equal to half the sum of $f(0)$ and $f(2\pi)$ both when $x=0$ and when $x=2\pi$.* Thus, in example 1 of § 52, every term of the series in equation (ii) (except the first, which is equal to π) is equal to 0 when $x=0$ and when $x=2\pi$; the value of the series for these two values of x is therefore π. But when $x=0$ the function x is 0, and when $x=2\pi$ the function x is 2π. For these values of x, therefore, the series and the function are not equal; the series is equal to π, which is half the sum of the values 0 and 2π of the function.

54. Change of Origin. It is often convenient to suppose that x varies from $-\pi$ to π, instead of from 0 to 2π; the range of x is still 2π. Graphically considered, the origin of coordinates is simply shifted to the point $(\pi, 0)$, and therefore x is replaced by $x+\pi$. It is, however, simpler to work out the values of the coefficients afresh.

$$\text{Let} \qquad f(x)=A_0+\sum_{r=1}^{r=\infty} A_r\cos rx+\sum_{r=1}^{r=\infty} B_r\sin rx \quad \ldots\ldots\ldots\ldots(1)$$

where x now varies from $-\pi$ to π.

To find the coefficients proceed exactly as before, but integrate from $-\pi$ to π. We find

$$\pi A_r=\int_{-\pi}^{\pi} f(x)\cos rx\,dx \ldots(2); \qquad \pi B_r=\int_{-\pi}^{\pi} f(x)\sin rx\,dx \ldots(3);$$

$$2\pi A_0=\int_{-\pi}^{\pi} f(x)\,dx. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Note that if $f(-\pi)\neq f(\pi)$, the value of the series in (1) is equal to $\frac{1}{2}\{f(-\pi)+f(\pi)\}$ both when $x=-\pi$ and when $x=\pi$.

Example. Suppose $f(x)=x$. It will be readily found that

$$A_0=0,\ A_r=0,\ B_r=-2\cos r\pi/r, \text{ so that}$$

$$B_r=\frac{2}{r} \text{ if } r \text{ is odd, } B_r=-\frac{2}{r} \text{ if } r \text{ is even.}$$

Hence

$$x=2\left(\frac{\sin x}{1}-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}-\frac{\sin 4x}{4}+\ldots\right).$$

When $x=-\pi$ the function x is $-\pi$, and when $x=\pi$ the function x is π; the value of the series should therefore, both when $x=-\pi$ and when $x=\pi$, be $\frac{1}{2}(-\pi+\pi)$, that is zero, and this is obviously the case.

This series for x is not the same as that in § 52. The reason is, that in the two series *the range of x is different*. In § 52 we must have $0<x<2\pi$, while here we have $-\pi<x<\pi$. This series may, however, be derived from that in § 52 by replacing x in equation (ii) of that article by $x+\pi$.

55. Sine Series and Cosine Series. The value of πA_r given by equation (2), § 54, may be written

$$\pi A_r=\int_{-\pi}^{0} f(x)\cos rx\,dx+\int_{0}^{\pi} f(x)\cos rx\,dx.$$

In the first of these integrals put $-u$ for x. Then $u=\pi$ when $x=-\pi$, and $u=0$ when $x=0$; also $f(x)=f(-u)$, $\cos rx=\cos ru$, $dx=-du$. Therefore

$$\int_{-\pi}^{0} f(x)\cos rx\,dx = -\int_{\pi}^{0} f(-u)\cos ru\,du = \int_{0}^{\pi} f(-u)\cos ru\,du.$$

In the last integral we may write x in place of u; the value of πA_r now becomes

$$\pi A_r = \int_0^{\pi} f(-x)\cos rx\,dx + \int_0^{\pi} f(x)\cos rx\,dx$$
$$= \int_0^{\pi}\{f(x)+f(-x)\}\cos rx\,dx. \quad\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

The same transformation gives for πB_r, $2\pi A_0$ of § 54

$$\pi B_r = \int_0^{\pi}\{f(x)-f(-x)\}\sin rx\,dx, \quad\ldots\ldots\ldots\ldots(6)$$

$$2\pi A_0 = \int_0^{\pi}\{f(x)+f(-x)\}dx. \quad\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

Now, suppose $f(x)$ to be an even function of x, that is, suppose that $f(-x)=f(x)$ (the function changing neither in sign nor in numerical value when x changes sign); then, the integrand in (6) is zero and B_r is zero, while the integrands in (5) and (7) become $2f(x)\cos rx$ and $2f(x)$ respectively. Therefore, dividing each member by 2,

$$\frac{\pi}{2}A_r = \int_0^{\pi} f(x)\cos rx\,dx \ldots(8); \quad \pi A_0 = \int_0^{\pi} f(x)\,dx. \ldots\ldots\ldots\ldots(9)$$

Next, suppose $f(x)$ to be an odd function of x, that is, suppose that $f(-x)=-f(x)$ (the function changing in sign but not in numerical value when x changes sign); A_0 and A_r are now zero while for B_r we have

$$\frac{\pi}{2}B_r = \int_0^{\pi} f(x)\sin rx\,dx. \quad\ldots\ldots\ldots\ldots\ldots\ldots(10)$$

It should be noticed that the values of A_r in (8) and A_0 in (9) imply that $f(x)$ is given by a series of the form

$$f(x) = A_0 + \sum_{r=1}^{r=\infty} A_r \cos rx \quad\ldots\ldots\ldots\ldots\ldots\ldots(11)$$

in which there are no sine terms. The series (11) is a **cosine series,** and $f(x)$ need only be defined for a half period, that is, from $x=0$ to $x=\pi$. The value of the series when x lies between 0 and $-\pi$ is the same as for the corresponding value of x between 0 and π. The value of the series when $x=0$ is $f(0)$, and when $x=\pi$ is $f(\pi)$; that is, the series and the function $f(x)$ are equal both when $x=0$ and when $x=\pi$.

Similarly B_r, as given by (10), is the coefficient of $\sin rx$ in the series

$$f(x)=\sum_{r=1}^{r=\infty} B_r \sin rx \text{ ....(12)}$$

in which there are no cosine terms; the series (12) is a **sine series.** The value of the series when $x=0$ is obviously 0, and when $x=\pi$ is also 0; the value of the function $f(x)$ is not necessarily zero for $x=0$ and $x=\pi$.

We may note that A_r may be found from (11) by multiplying by $\cos rx$ and integrating from 0 to π, and B_r may be found from (12) by multiplying by $\sin rx$ and integrating from 0 to π. In practice it is usually better to find the coefficients in this way than to apply the formulae which give their value. The method, however, by which the results in this article and in § 54 have been deduced has the advantage of showing their connection with the theorem of § 52. When the function is defined for the whole period there is *only one series* that represents it; but when the function is only defined for the half period, there are two essentially different series for the function, namely the sine series and the cosine series. As a matter of fact, when the function is only defined for the half period we can find many series to represent it, but the sine and the cosine series are the only two that are of much importance.

56. Worked Examples. We shall now work out some examples illustrating the various formulae. The student should always keep the following peculiarities in mind.

(i) If the function $f(x)$ is discontinuous when $x=a$ (in the way stated in § 53, (i)) the value of the series when $x=a$ is half the sum of the two values of $f(x)$ when $x=a$.

(ii) If the range of x is from 0 to 2π (§ 52) the value of

the series, both when $x=0$ and when $x=2\pi$, is

$$\tfrac{1}{2}\{f(0)+f(2\pi)\}.$$

Therefore, *unless* $f(2\pi)=f(0)$, the value of the series and the value of the function are not the same, either when $x=0$ or when $x=2\pi$.

(iii) If the range of x is from $-\pi$ to π (§ 54) the value of the series, both when $x=-\pi$ and when $x=\pi$, is $\tfrac{1}{2}\{f(-\pi)+f(\pi)\}$. Therefore, *unless* $f(-\pi)=f(\pi)$, the value of the series and the value of the function are not the same, either when $x=-\pi$ or when $x=\pi$.

(iv) If the range of x is from 0 to π, *the function is only defined for half the period.* In this case the function may be represented *either* by a sine series *or* by a cosine series (§ 55). The sine series is zero both when $x=0$ and when $x=\pi$; therefore the sine series will not be equal to the function for these values of x, unless the function is itself zero for these values. The cosine series, on the other hand, is equal to the function both when $x=0$ and when $x=\pi$.

Example 1. Let $f(x)=x$ from $x=0$ to $x=\pi/2$, but let $f(x)=\pi-x$ from $x=\pi/2$ to $x=\pi$. Find a sine series for $f(x)$.

The graph of $f(x)$ is the broken line OAB (Fig. 28); along OA the ordinate is equal to x, but along AB the ordinate is equal to $\pi-x$.

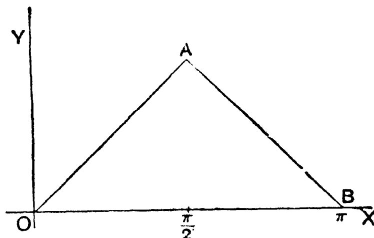


Fig. 28.

In evaluating the integrals in a case like this we must divide the range of integration into parts corresponding to the different expressions for $f(x)$. In the present case we have

$$\frac{\pi}{2}B_r=\int_0^{\frac{\pi}{2}} x\sin rx\,dx+\int_{\frac{\pi}{2}}^{\pi}(\pi-x)\sin rx\,dx.$$

But, integrating by parts, we find

$$\int x\sin rx\,dx=-\frac{x\cos rx}{r}+\frac{\sin rx}{r^2},$$

and therefore

$$\int_0^{\frac{\pi}{2}} x \sin rx dx = -\frac{\pi}{2r}\cos\frac{r\pi}{2} + \frac{1}{r^2}\sin\frac{r\pi}{2},$$

$$\int_{\frac{\pi}{2}}^{\pi} (\pi - x)\sin rx dx = \int_{\frac{\pi}{2}}^{\pi} \pi \sin rx dx - \int_{\frac{\pi}{2}}^{\pi} x \sin rx dx$$

$$= \frac{\pi}{2r}\cos\frac{r\pi}{2} + \frac{1}{r^2}\sin\frac{r\pi}{2}.$$

Combining these results, we obtain

$$\frac{\pi}{2}B_r = \frac{2}{r^2}\sin\frac{r\pi}{2}, \text{ or, } B_r = \frac{4}{\pi r^2}\sin\frac{r\pi}{2},$$

so that $B_r = 0$ when r is even, but $B_r = \pm 4/\pi r^2$ when r is odd. The sign is $+$ when r is 1, 5, 9, ... and $-$ when r is 3, 7, 11, ... Hence

$$f(x) = \frac{4}{\pi}\left(\frac{\sin x}{1^2} - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \frac{\sin 7x}{7^2} + \dots\right). \quad \text{............(i)}$$

The series in (i) is therefore equal to x from $x=0$ to $x=\pi/2$, but is equal to $\pi - x$ from $x=\pi/2$ to $x=\pi$; the series is equal to the function both when $x=0$ and when $x=\pi$. When x is negative,

$$\sin rx = -\sin(-rx);$$

therefore, when x is negative, the series is equal to x from $x=0$ to $x=-\pi/2$, but is equal to $-(\pi+x)$ from $x=-\pi/2$ to $-\pi$. For instance, when $x=-2\pi/3$ the series is equal to $-(\pi-2\pi/3)$ or $-\pi/3$.

From (i) we deduce a remarkable series by putting $x=\pi/2$. When $x=\pi/2$ the function $f(x)$ is also $\pi/2$, and therefore

$$\frac{\pi}{2} = \frac{4}{\pi}\left(\frac{1}{1^2} - \frac{-1}{3^2} + \frac{1}{5^2} - \frac{-1}{7^2} + \dots\right),$$

or,

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots \text{ to infinity.} \quad \text{..................(ii)}$$

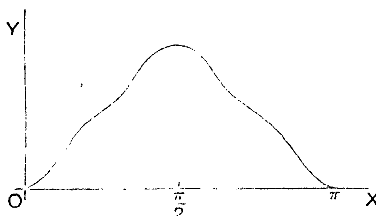


Fig. 29.

The student should test, by plotting the graph of the series, that its graph coincides with the broken line; a few terms give a fair approximation. In Fig. 29 we show the graph of

$$v = \frac{4}{\pi}\left(\frac{\sin x}{1^2} - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \frac{\sin 7x}{7^2}\right).$$

Example 2. Find a cosine series for x^2.

Integrating by parts, we find

$$\frac{\pi}{2}A_r=\int_0^\pi x^2\cos rx\,dx=\left[\frac{x^2\sin rx}{r}+\frac{2x\cos rx}{r^2}-\frac{2\sin rx}{r^3}\right]_0^\pi,$$

and therefore

$$\frac{\pi}{2}A_r=\frac{2\pi\cos r\pi}{r^2},\quad A_r=\frac{4\cos r\pi}{r^2}.$$

Also, $$\pi A_0=\int_0^\pi x^2dx=\frac{\pi^3}{3},\quad A_0=\frac{\pi^2}{3}.$$

Hence,

$$x^2=\frac{\pi^2}{3}-4\left(\frac{\cos x}{1^2}-\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}-\frac{\cos 4x}{4^2}+\dots\right). \quad\text{............(i)}$$

Equation (i) is true both when $x=0$ and when $x=\pi$; it holds in fact from $x=-\pi$ to $x=\pi$, the values $-\pi$ and π included.

In (i) put first $x=0$, then $x=\pi$; we find

$$\frac{\pi^2}{12}=\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\dots\text{ to infinity,} \quad\text{..................(ii)}$$

$$\frac{\pi^2}{6}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\dots\text{ to infinity.} \quad\text{.................(iii)}$$

By addition of (ii) and (iii) we deduce series (ii) of example 1.

Example 3. Find a sine series that will be equal to 1 from $x=0$ to $x=\pi/2$, but equal to 0 from $x=\pi/2$ to $x=\pi$.

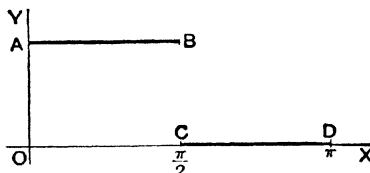


Fig. 30.

The graph of the function $f(x)$ consists in this case of the two straight lines AB, CD (Fig. 30); AB is parallel to the x-axis at a distance 1 above the axis, while CD is a part of the x-axis. The length of each line is $\pi/2$.

$$\frac{\pi}{2}B_r=\int_0^{\frac{\pi}{2}}1\,.\,\sin rx\,dx+\int_{\frac{\pi}{2}}^{\pi}0\,.\,\sin rx\,dx=\left[-\frac{\cos rx}{r}\right]_0^{\frac{\pi}{2}}$$

because the second integrand, and therefore the second integral, is zero. Therefore

$$B_r=\frac{2}{\pi r}\left(1-\cos\frac{r\pi}{2}\right);$$

$B_r=0$, when r is 0, 4, 8, ... $4n$, $\quad$ n being any integer;

$B_r=\frac{2}{\pi r}$, when r is 1, 3, 5, ... $2n-1$, ...

$B_r=\frac{4}{\pi r}$, when r is 2, 6, 10, ... $4n-2$, ...

Hence we have

$$f(x)=\frac{2}{\pi}\left(\frac{\sin x}{1}+\frac{2\sin 2x}{2}+\frac{\sin 3x}{3}+\frac{\sin 5x}{5}+\frac{2\sin 6x}{6}+\dots\right). \quad \text{......(i)}$$

When $x=\pi/2$, the series in (i) is equal to

$$\frac{2}{\pi}\left(\tfrac{1}{1}-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\dots\right)=\frac{2}{\pi}\times\frac{\pi}{4}=\tfrac{1}{2},$$

by § 52, example 2. The two ordinates at $x=\pi/2$ are CB, which is 1, and the ordinate of CD, which is 0. The half sum of these is $\frac{1}{2}$, so that the statement of the value of the series at a point of discontinuity is verified for this case.

The student will find it an interesting exercise to draw the graph of the series in (i) and to notice how, as more and more terms of the series are taken, the graph near $x=\pi/2$ becomes steeper and steeper.

Example 4. Find a series for $f(x)$ when $f(x)=0$ from $x=-\pi$ to $x=0$, and $f(x)=x$ from $x=0$ to $x=\pi$.

In this example the function is given for the whole period, and therefore there can be only one series; we use the formulae of § 54, decomposing the integrals to suit the different values of $f(x)$ in the two parts of the range.

$$\pi A_r=\int_{-\pi}^{0}0\,.\cos rx\,dx+\int_0^{\pi}x\cos rx\,dx=\frac{\cos r\pi-1}{r^2},$$

$$\pi B_r=\int_{-\pi}^{0}0\,.\sin rx\,dx+\int_0^{\pi}x\sin rx\,dx=-\frac{\pi\cos r\pi}{r},$$

$$2\pi A_0=\int_{-\pi}^{0}0\,.\,dx+\int_0^{\pi}x\,dx=\frac{\pi^2}{2}.$$

Therefore, $A_0=\pi/4$; $A_r=0$ when r is even, but $-2/\pi r^2$ when r is odd; $B_r=1/r$ when r is odd, but $-1/r$ when r is even. Hence

$$f(x)=\frac{\pi}{4}-\frac{2}{\pi}\left(\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+\frac{\cos 7x}{7^2}+\dots\right)$$
$$+\left(\frac{\sin x}{1}-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}-\frac{\sin 4x}{4}+\dots\right). \quad \text{..................(i)}$$

Both when $x=-\pi$ and when $x=\pi$ the series in (i) is equal to

$$\frac{\pi}{4}+\frac{2}{\pi}\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots\right)=\frac{\pi}{4}+\frac{2}{\pi}\cdot\frac{\pi^2}{8}=\frac{\pi}{2},$$

by example 1 (ii). But when $x=-\pi$, $f(x)=0$ and when $x=\pi$, $f(x)=\pi$; so that $\frac{1}{2}\{f(-\pi)+f(\pi)\}=\frac{1}{2}\pi$. We thus verify for this case the statement (iii) at the beginning of this article.

57. Period any given Number. Up to this stage we have supposed the period to be 2π. A function $f(x)$ is said to have the period a if $f(x \pm na) = f(x)$, where n is any positive integer. If the function to be represented has the period a, or if, without being periodic, it is to be represented by a Fourier series of period a, the angle will be $2\pi x/a$ instead of x. For, if n is any integer

$$\sin\frac{2\pi(x\pm na)}{a} = \sin\frac{2\pi x}{a}, \quad \cos\frac{2\pi(x\pm na)}{a} = \cos\frac{2\pi x}{a}.$$

The form of the series is now

$$f(x) = A_0 + \sum_{r=1}^{r=\infty} A_r \cos\frac{2\pi rx}{a} + \sum_{r=1}^{r=\infty} B_r \sin\frac{2\pi rx}{a}. \quad \ldots\ldots\ldots(1)$$

To obtain the values of A_0, A_r, B_r we proceed as before, but the limits of the integrals will be 0 and a, instead of 0 and 2π, $-\frac{1}{2}a$ and $\frac{1}{2}a$, instead of $-\pi$ and π, 0 and $\frac{1}{2}a$ instead of 0 and π.

For example, the formulae (5), (6), (7) of § 52 are replaced by

$$\frac{a}{2}A_r = \int_0^a f(x)\cos\frac{2\pi rx}{a}dx \;\ldots(2); \quad \frac{a}{2}B_r = \int_0^a f(x)\sin\frac{2\pi rx}{a}dx \ldots(3);$$

$$aA_0 = \int_0^a f(x)dx.$$

In fact, in the coefficients A_0, A_r, B_r in the previous formulae, the former half period π is replaced by the new half period $a/2$ while the angle x is replaced by $2\pi x/a$. If $a = 2\pi$, the new formulae reduce to the old.

All that has been said as to discontinuities, value of series when $x = -\pi$ and when $x = \pi$, etc., holds good; thus, the value of the series in (1) above, when $x = 0$ and when $x = a$, is $\frac{1}{2}\{f(0) + f(a)\}$ if the range of x is from 0 to a.

Example. Find a sine series that will represent 1 from $x=0$ to $x=c$.

This problem corresponds to the previous case of a sine series in which x varies from 0 to π; c is the half period, so that here $a = 2c$. Putting therefore $2c$ for a in (1), and retaining only the sine series, we write

$$f(x) = \sum_{r=1}^{r=\infty} B_r \sin\frac{\pi rx}{c}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(i)$$

Now multiply both sides of (i) by $\sin(\pi rx/c)$ and integrate from 0 to c. After integration all the terms on the right of (i) vanish except that containing B_r, which gives

$$\int_0^c B_r \sin^2\frac{\pi rx}{c}\,dx = \tfrac{1}{2}B_r\int_0^c\left(1-\cos\frac{2\pi rx}{c}\right)dx = \frac{c}{2}B_r.$$

Remembering that $f(x)=1$, we see that the integral of the left side of (i) is

$$\int_0^c 1\,.\,\sin\frac{\pi rx}{c}\,dx = \left[-\frac{c}{\pi r}\cos\frac{\pi rx}{c}\right]_0^c = \frac{c}{\pi r}(1-\cos r\pi),$$

and therefore $$B_r = \frac{2}{\pi r}(1-\cos r\pi).$$

Hence $$1 = \frac{4}{\pi}\left(\sin\frac{\pi x}{c} + \tfrac{1}{3}\sin\frac{3\pi x}{c} + \tfrac{1}{5}\sin\frac{5\pi x}{c} + \dots\right). \quad \text{............(ii)}$$

The series is zero both when $x=0$ and when $x=c$, and is therefore not equal to the function for these values of x.

EXERCISES. XVI.

In each of the examples 1–4 twelve equidistant ordinates of a curve are given, corresponding to the values 0, $\pi/6$, $2\pi/6$ ... of the abscissa. Analyse the curve into its harmonic components.

1. 32, 60, 64, 20, 15, 49, 20, −30, −38, 40, 47, 38.

2. 12, 38, 12, −23, −32, 20, −21, −40, −12, 34, 56, 45.

3. 6, 10, 9·8, 6, 2·3, −1, −3·3, −5·4, −7·7, −8·8, −6, −1.

4. 19·4, 19·6, 14·5, 4·5, −6·1, −12·4, −14·8, −14·6, −12·9, −3·9, 8·9, 15·8.

5. Show that the quantities a_0, a_1, ..., b_1, b_2, ... of § 50 satisfy the equations

$$\tfrac{1}{2}a_r = \sum_{s=0}^{s=6} A_s \cos sr\theta, \quad \tfrac{1}{2}b_r = \sum_{s=1}^{s=5} B_s \sin sr\theta$$

for $r=1, 2, 3, 4, 5$; while for $r=0, 6$

$$a_0 = \sum_{s=0}^{s=6} A_s, \quad a_6 = \sum_{s=0}^{s=6} A_s \cos s\pi.$$

Hence, show that a_0, $\frac{1}{2}a_1$, $\frac{1}{2}a_2$, $\frac{1}{2}a_3$, $\frac{1}{2}a_4$, $\frac{1}{2}a_5$, a_6 may be found from A_0, A_1, ... A_6 by the same scheme as $12A_0$, $6A_1$, $6A_2$, $6A_3$, $6A_4$, $6A_5$, $12A_6$ are found from a_0, a_1, ... a_6; and that the quantities $\frac{1}{2}b_r$ may be found from the quantities B_r by the same scheme as the quantities $6B_r$ are found from the quantities b_r.

Apply these results to test the values found in examples 1–4, noting that the values $6A_0$, $6A_1$, ... $6A_6$, $6B_1$, $6B_2$, ... may be used, the calculated values being then $6a_0$, $3a_1$, ... $3a_5$, $6a_6$, $3b_1$, $3b_2$, ... $3b_5$.

6. If $4p$ equidistant ordinates $y_0, y_1, y_2, \ldots$ are given corresponding to the values $0, \theta, 2\theta, \ldots$ of x, where $\theta = 2\pi/4p = \pi/2p$; and if

$$y = A_0 + \sum_{r=1}^{r=2p} A_r \cos rx + \sum_{r=1}^{r=2p-1} B_r \sin rx$$

show that, with the same notations as in § 50,

$$2pA_r = \sum_{s=0}^{s=p-1} c'_s \cos rs\theta, \quad 2pA_{2p-r} = \sum_{s=0}^{s=p-1} c'_s \cos s\pi \cos rs\theta,$$

$$2pB_r = \sum_{s=1}^{s=p} d_s \sin rs\theta, \quad 2pB_{2p-r} = \sum_{s=1}^{s=p} d_s \cos(s-1)\pi \sin rs\theta$$

when r is odd; but

$$2pA_r = \sum_{s=0}^{s=p} c_s \cos rs\theta, \quad 2pA_{2p-r} = \sum_{s=0}^{s=p} c_s \cos s\pi \cos rs\theta,$$

$$2pB_r = \sum_{s=1}^{s=p-1} d'_s \sin rs\theta, \quad 2pB_{2p-r} = \sum_{s=1}^{s=p-1} d'_s \cos(s-1)\pi \sin rs\theta$$

when r is even. When $r=0$, $2p$, the values are

$$4pA_0 = \sum_{s=0}^{s=p} c_s, \qquad 4pA_{2p} = \sum_{s=0}^{s=p} c_s \cos s\pi.$$

[The above equations are the most convenient for calculations. If $4p=36$ the angles that occur are, in degrees, 0°, 10°, 20°, ... 90° or may be reduced to these. Runge in the article referred to in § 48 gives a scheme for 36 ordinates, analogous to that of § 51 for 12 ordinates, though necessarily more complicated. Tests analogous to those given in example 5 can be readily established for the case of $4p$ equidistant ordinates.]

7. Find a cosine series for x. Test that the series is zero when $x=0$, and is equal to π when $x=\pi$.

8. Show, by expanding $\sin x$ in a cosine series that

$$\sin x = \frac{2}{\pi}\left(1 - \frac{2\cos 2x}{1 \cdot 3} - \frac{2\cos 4x}{3 \cdot 5} - \frac{2\cos 6x}{5 \cdot 7} - \ldots\right).$$

Verify that the series is zero both when $x=0$ and when $x=\pi$. What function does the series represent when x lies between 0 and $-\pi$?

9. Find a cosine series that is equal to 1 from $x=0$ to $x=\pi/2$, and equal to 0 from $x=\pi/2$ to $x=\pi$. What is the value of the series when x has the values $0, \pi/2, \pi$?

10. Find a series that is equal to -1 from $x=-\pi$ to $x=-\pi/2$, equal to 0 from $x=-\pi/2$ to $x=0$, equal to 1 from $x=0$ to $x=\pi/2$, and equal to 0 from $x=\pi/2$ to $x=\pi$. What is the value of the series when x has the values $-\pi, -\pi/2, 0, \pi/2, \pi$?

11. Find a sine series that is equal to x from $x=0$ to $x=\pi/3$, equal to $\pi/3$ from $x=\pi/3$ to $x=2\pi/3$, and equal to $\pi-x$ from $x=2\pi/3$ to $x=\pi$. What does the series represent for values of x from $x=0$ to $x=-\pi$?

12. Find a sine series that is equal to 1.

13. Find a series that represents x^2 from $x=-\pi$ to $x=\pi$. Find also a sine series for x^2 (from $x=0$ to $x=\pi$).

14. Show that, if a is a fraction,

$$\cos ax=\frac{2a\sin a\pi}{\pi}\left\{\frac{1}{2a^2}-\frac{\cos x}{a^2-1^2}+\frac{\cos 2x}{a^2-2^2}-\frac{\cos 3x}{a^2-3^2}+\dots\right\}.$$

15. Find, by using example 14, the sum of the infinite series

$$\frac{1}{a}-\left(\frac{1}{a-1}+\frac{1}{a+1}\right)+\left(\frac{1}{a-2}+\frac{1}{a+2}\right)-\left(\frac{1}{a-3}+\frac{1}{a+3}\right)+\dots$$

where a is a fraction.

16. Show that, if a is a fraction,

$$\sin ax=\frac{2\sin a\pi}{\pi}\left\{\frac{\sin x}{1^2-a^2}-\frac{2\sin 2x}{2^2-a^2}+\frac{3\sin 3x}{3^2-a^2}-\frac{4\sin 4x}{4^2-a^2}+\dots\right\}.$$

17. Find, by using example 16, the sum of the infinite series

$$\frac{1}{1-a}+\frac{1}{1+a}-\left(\frac{1}{3-a}+\frac{1}{3+a}\right)+\left(\frac{1}{5-a}+\frac{1}{5+a}\right)-\left(\frac{1}{7-a}+\frac{1}{7+a}\right)+\dots$$

where a is a fraction.

18. Deduce from example 8 the sum of the infinite series

$$\tfrac{1}{1}+\tfrac{1}{3}-\tfrac{1}{5}-\tfrac{1}{7}+\tfrac{1}{9}+\tfrac{1}{11}-\tfrac{1}{13}-\tfrac{1}{15}+\dots.$$

19. Prove, by integrating both sides of equation (ii), example 1, § 52, that

$$\frac{(x-\pi)^2}{4}-\frac{\pi^2}{12}=\frac{\cos x}{1}+\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}+\dots.$$

20. Deduce the equation in example 16 by differentiating that in example 14.

CHAPTER XII.

INVERSE CIRCULAR FUNCTIONS. LOGARITHMIC AND EXPONENTIAL FUNCTIONS.

58. Inverse Circular Functions. It is necessary to remember the restriction on the range of the angles denoted by the symbols $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, $\cot^{-1}x$.

The range of $\sin^{-1}x$ and of $\tan^{-1}x$ is from $-\pi/2$ to $\pi/2$, inclusive of these angles; the range of $\cos^{-1}x$ and of $\cot^{-1}x$ is from 0 to π, inclusive of these angles. For example,

$$\sin^{-1}(-\tfrac{1}{2})=-\frac{\pi}{6},\ \cos^{-1}(-\tfrac{1}{2})=\frac{2\pi}{3},\ \sin^{-1}(-1)=-\frac{\pi}{2},\ \cos^{-1}(-1)=\pi,$$

$$\tan^{-1}(-1)=-\frac{\pi}{4},\ \cot^{-1}(-1)=\frac{3\pi}{4},\ \tan^{-1}(-\infty)=-\frac{\pi}{2},\ \cot^{-1}(-\infty)=\pi.$$

The argument of these functions occurs so frequently in the combination x/a that it is well to have the derivatives for that combination; we therefore state the derivatives for both arguments, x and x/a. We suppose a to be positive.

$$\text{I.}\quad \mathbf{D_x\sin^{-1}x=\frac{1}{\sqrt{(1-x^2)}}};\quad \mathbf{D_x\sin^{-1}\left(\frac{x}{a}\right)=\frac{1}{\sqrt{(a^2-x^2)}}}.$$

Take the second form, and let $y=\sin^{-1}(x/a)$. Then

$$x=a\sin y,\quad \frac{dx}{dy}=a\cos y,\quad \frac{dy}{dx}=\frac{1}{a\cos y}.$$

But
$$\cos^2 y=1-\sin^2 y=1-\frac{x^2}{a^2}=\frac{a^2-x^2}{a^2},$$

so that $a\cos y=\sqrt{(a^2-x^2)}$; the + sign must be given to the root, because y lies between $-\pi/2$ and $\pi/2$, and $\cos y$ is therefore positive. Hence,

$$\frac{dy}{dx}=\frac{1}{\sqrt{(a^2-x^2)}}.$$

II. $D_x\cos^{-1}x = \dfrac{-1}{\sqrt{(1-x^2)}}$; $D_x\cos^{-1}\left(\dfrac{x}{a}\right) = \dfrac{-1}{\sqrt{(a^2-x^2)}}$.

Let $y = \cos^{-1}(x/a)$, so that $x = a\cos y$. Then

$$\frac{dx}{dy} = -a\sin y,\quad \frac{dy}{dx} = \frac{-1}{a\sin y} = \frac{-1}{\sqrt{(a^2-x^2)}}.$$

The + sign must be given to the root, because y lies between 0 and π, and therefore $\sin y$ is positive.

III. $D_x\tan^{-1}x = \dfrac{1}{1+x^2}$; $D_x\tan^{-1}\left(\dfrac{x}{a}\right) = \dfrac{a}{a^2+x^2}$.

Let $y = \tan^{-1}(x/a)$, so that $x = a\tan y$. Then

$$\frac{dx}{dy} = a\sec^2 y = a(1+\tan^2 y) = \frac{a^2+x^2}{a}$$

and therefore $$\frac{dy}{dx} = \frac{a}{a^2+x^2}.$$

In a similar way it is proved that

IV. $D_x\cot^{-1}x = \dfrac{-1}{1+x^2}$; $D_x\cot^{-1}\left(\dfrac{x}{a}\right) = \dfrac{-a}{a^2+x^2}$.

From these results we at once obtain the integrals:

V. $\displaystyle\int\frac{dx}{\sqrt{(1-x^2)}} = \sin^{-1}x$, $\displaystyle\int\frac{dx}{\sqrt{(a^2-x^2)}} = \sin^{-1}\left(\frac{x}{a}\right)$,

or $= -\cos^{-1}x$; or $= -\cos^{-1}\left(\dfrac{x}{a}\right)$.

VI. $\displaystyle\int\frac{dx}{1+x^2} = \tan^{-1}x$, $\displaystyle\int\frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$,

or $= -\cot^{-1}x$; or $= -\dfrac{1}{a}\cot^{-1}\left(\dfrac{x}{a}\right)$.

The following important integral is proved in § 45, example 6:

VII. $\displaystyle\int\sqrt{(a^2-x^2)}\,dx = \tfrac{1}{2}x\sqrt{(a^2-x^2)} + \tfrac{1}{2}a^2\sin^{-1}\left(\frac{x}{a}\right)$.

Note that $\sin^{-1}x$ and $-\cos^{-1}x$ differ by a constant; for

$$\sin^{-1}x - (-\cos^{-1}x) = \sin^{-1}x + \cos^{-1}x = \pi/2.$$

The two functions may therefore be integrals of the same function. A similar remark applies to $\tan^{-1}x$ and $-\cot^{-1}x$.

59. Worked Examples. For elementary work the chief value of the inverse circular functions is that they furnish integrals of functions of the form

$$\frac{1}{\sqrt{(c+bx-x^2)}} \text{ and } \frac{1}{x^2+bx+c},$$

the factors of x^2+bx+c being *imaginary*, like those of x^2+a^2. We illustrate by the following examples:

Example 1. Evaluate $\displaystyle\int\frac{dx}{\sqrt{\{c^2-(x+b)^2\}}}$.

Let $x+b=u$; then $dx=du$, and

$$\int\frac{dx}{\sqrt{\{c^2-(x+b)^2\}}}=\int\frac{du}{\sqrt{(c^2-u^2)}}=\sin^{-1}\left(\frac{u}{c}\right).$$

Replacing u by $x+b$, we find that

$$\int\frac{dx}{\sqrt{\{c^2-(x+b)^2\}}}=\sin^{-1}\left(\frac{x+b}{c}\right).$$

Example 2. Evaluate $\displaystyle\int\frac{dx}{\sqrt{(4+5x-3x^2)}}$.

The first step is to reduce $4+5x-3x^2$ to the form of the difference of two squares, that is, to the form $c^2-(x+b)^2$; of course, when the coefficient of x^2 is not -1 there will be a constant factor.

In the present case we have

$$4+5x-3x^2=3\left(\frac{4}{3}+\frac{5}{3}x-x^2\right)=3\left\{\frac{73}{36}-\left(x-\frac{5}{6}\right)^2\right\},$$

$$\sqrt{(4+5x-3x^2)}=\sqrt{3}\,.\,\sqrt{\left\{\left(\frac{\sqrt{73}}{6}\right)^2-\left(x-\frac{5}{6}\right)^2\right\}},$$

so that $$\int\frac{dx}{\sqrt{(4+5x-3x^2)}}=\frac{1}{\sqrt{3}}\int\frac{dx}{\sqrt{\left\{\left(\frac{\sqrt{73}}{6}\right)^2-\left(x-\frac{5}{6}\right)^2\right\}}}.$$

In example 1 put $\sqrt{73}/6$ for c and $-5/6$ for b, and we find that the integral is equal to

$$\frac{1}{\sqrt{3}}\sin^{-1}\left(\frac{x-5/6}{\sqrt{73}/6}\right)=\frac{1}{\sqrt{3}}\sin^{-1}\left(\frac{6x-5}{\sqrt{73}}\right).$$

Example 3. Evaluate $\displaystyle\int\frac{dx}{(x+b)^2+c^2}$.

Put u for $x+b$, and the integral becomes

$$\int\frac{du}{u^2+c^2}=\frac{1}{c}\tan^{-1}\left(\frac{u}{c}\right)=\frac{1}{c}\tan^{-1}\left(\frac{x+b}{c}\right).$$

Example 4. Evaluate $\int \frac{dx}{5x^2-7x+8}$.

Arrange $5x^2-7x+8$ as the sum of two squares; we find

$$5x^2-7x+8=5\left(x^2-\frac{7}{5}x+\frac{8}{5}\right)=5\left\{\left(x-\frac{7}{10}\right)^2+\left(\frac{\sqrt{111}}{10}\right)^2\right\},$$

so that $$\int \frac{dx}{5x^2-7x+8}=\frac{1}{5}\int \frac{dx}{\left(x-\frac{7}{10}\right)^2+\left(\frac{\sqrt{111}}{10}\right)^2}$$

which, by example 4, is equal to

$$\frac{1}{5}\cdot\frac{10}{\sqrt{111}}\tan^{-1}\left(\frac{x-7/10}{\sqrt{111}/10}\right)=\frac{2}{\sqrt{111}}\tan^{-1}\left(\frac{10x-7}{\sqrt{111}}\right).$$

Note that $$\frac{1}{c}=1\div\frac{\sqrt{111}}{10}=\frac{10}{\sqrt{111}}.$$

If the constant term c^2 (in example 3) is a fraction, equal to m^2/n^2 say, the factor $1/c$ is n/m; the student should from the first practise writing it as n/m and not as $1\div(m/n)$.

The next example illustrates a method of obtaining a power series for a function that is sometimes useful.

Example 5. Show that, if x^2 is less than or equal to 1,

$$\tan^{-1}x=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\dots \text{ to infinity.}$$

Dividing 1 by $1+u^2$ we find

$$\frac{1}{1+u^2}=1-u^2+u^4-u^6+\dots \pm u^{2n-2}\mp\frac{u^{2n}}{1+u^2}. \quad\dots\dots\dots\dots\dots \text{(i)}$$

Integrate both sides of (i) from 0 to x; therefore

$$\tan^{-1}x=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\dots\pm\frac{x^{2n-1}}{2n-1}\mp\int_0^x\frac{u^{2n}\,du}{1+u^2}. \quad\dots\dots\dots\dots \text{(ii)}$$

Now, $u^{2n}/(1+u^2)$ is less than u^{2n}, and therefore the area between the graph of $u^{2n}/(1+u^2)$, the u-axis and the ordinate at $u=x$ is less than the corresponding area for the graph of u^{2n}. Hence

$$\int_0^x\frac{u^{2n}du}{1+u^2}<\int_0^x u^{2n}du \text{ or } \frac{x^{2n+1}}{2n+1}.$$

Further, if x^2 is not greater than 1, $x^{2n+1}/(2n+1)$ is not greater, numerically, than $1/(2n+1)$, and therefore becomes very small when n becomes very large. Hence we have

$$\tan^{-1}x=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\dots\pm\frac{x^{2n-1}}{2n-1}\mp\dots \quad\dots\dots\dots\dots\dots \text{(iii)}$$

where the error made by stopping at the term $x^{2n-1}/(2n-1)$ is less, numerically, than $x^{2n+1}/(2n+1)$.

If $x=1$ we find, since $\tan^{-1}1=\pi/4$,

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots, \qquad \text{(iv)}$$

the same series as in § 52, example 2.

The series (iv) is not very suitable for the calculation of π. A better series is obtained by using the identity (*Machin's* Formula)

$$\frac{\pi}{4}=4\tan^{-1}\left(\frac{1}{5}\right)-\tan^{-1}\left(\frac{1}{239}\right).$$

The values of $\tan^{-1}(1/5)$ and $\tan^{-1}(1/239)$ can be calculated from (iii) with very little labour.

In Exercises XVII., examples 37-40, the student will find a transformation that is useful in higher work but that may be passed over by beginners.

EXERCISES. XVII.

Differentiate the functions in examples 1–13.

1. $\sin^{-1}2x$.
2. $\sin^{-1}(1-x)$.
3. $\sin^{-1}\left(\frac{2x-1}{3}\right)$.
4. $\cos^{-1}(1-x)$.
5. $\sin^{-1}(ax+b)$.
6. $\tan^{-1}(1-x)$.
7. $\tan^{-1}\left(\frac{2x+1}{3}\right)$.
8. $x\sin^{-1}x$.
9. $x\sin^{-1}x+\sqrt{(1-x^2)}$.
10. $\frac{1}{2}x^2\sin^{-1}x-\frac{1}{4}\sin^{-1}x+\frac{1}{4}x\sqrt{(1-x^2)}$.
11. $\frac{1}{2}(x-1)\sqrt{(3+2x-x^2)}+2\sin^{-1}\left(\frac{x-1}{2}\right)$.
12. $x\tan^{-1}x$.
13. $(1+x^2)\tan^{-1}x-x$.

Integrate the functions in examples 14–30.

14. $\frac{1}{\sqrt{(3-x^2)}}$.
15. $\frac{1}{\sqrt{(9-4x^2)}}$.
16. $\frac{1}{\sqrt{(7-3x^2)}}$.
17. $\sqrt{(3-x^2)}$.
18. $\sqrt{(7-3x^2)}$.
19. $\sqrt{(b^2-a^2x^2)}$.
20. $\frac{1}{\sqrt{(6x-x^2)}}$.
21. $\sqrt{(6x-x^2)}$.
22. $\frac{1}{\sqrt{(ax-x^2)}}$.
23. $\sqrt{(ax-x^2)}$.
24. $\frac{1}{\sqrt{(3+4x-4x^2)}}$.
25. $\sqrt{(3+4x-4x^2)}$.
26. $\sqrt{(24x-9x^2-7)}$.
27. $\frac{1}{3+x^2}$.
28. $\frac{1}{a^2+b^2x^2}$.
29. $\frac{1}{x^2+x+1}$.
30. $\frac{1}{9x^2+24x+25}$.

Evaluate the definite integrals in examples 31–36.

31. $\displaystyle\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{(1-x^2)}}$.   32. $\displaystyle\int_0^2 \frac{dx}{\sqrt{(4-x^2)}}$.   33. $\displaystyle\int_0^a \frac{dx}{a^2+x^2}$.

34. $\displaystyle\int_1^2 \frac{dx}{\sqrt{(2x-x^2)}}$.   35. $\displaystyle\int_0^2 \frac{dx}{\sqrt{(2x-x^2)}}$.   36. $\displaystyle\int_0^{2a} \frac{dx}{\sqrt{(2ax-x^2)}}$

37. If $\tan \frac{1}{2}x = u$, show that

$$\sin x = \frac{2u}{1+u^2}, \quad \cos x = \frac{1-u^2}{1+u^2},$$

$$x = 2\tan^{-1}u, \quad dx = \frac{2du}{1+u^2}.$$

Then, show that

$$\int \frac{dx}{5+3\cos x} = \int \frac{du}{4+u^2} = \tfrac{1}{2}\tan^{-1}(\tfrac{1}{2}\tan \tfrac{1}{2}x),$$

$$\int_0^{\pi} \frac{dx}{5+3\cos x} = \tfrac{1}{2}\tan^{-1}(\infty) = \frac{\pi}{4}.$$

38. Show, by the method used in example 37, that if $a^2 > b^2$, and a positive

(i) $$\int \frac{dx}{a+b\cos x} = \frac{2}{\sqrt{(a^2-b^2)}}\tan^{-1}\left\{\sqrt{\left(\frac{a-b}{a+b}\right)} \cdot \tan \tfrac{1}{2}x\right\},$$

(ii) $$\int_0^{\pi} \frac{dx}{a+b\cos x} = \frac{\pi}{\sqrt{(a^2-b^2)}}.$$

Verify (i) by differentiation.

39. Show that, if $\tan \frac{1}{2}x = u$,

$$\int \frac{dx}{5+3\sin x} = \int \frac{2du}{5+6u+5u^2} = \tfrac{1}{2}\tan^{-1}\left(\frac{3+5\tan \frac{1}{2}x}{4}\right).$$

40. Prove that

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{dx}{5+3\sin x} = \frac{\pi}{4}.$$

60. Compound Interest Law. Suppose a principal of £P to earn interest at the rate of p per cent. per annum; let the interest be calculated at n equal intervals in each year and let it be added to the principal as soon as it is earned, so that the interest, as soon as it is earned, begins to bear interest. It is easy to prove that at the end of t years the principal will amount to

$$P\left(1+\frac{p}{100n}\right)^{nt}.$$

Let $$\frac{p}{100n}=\frac{1}{m}, \text{ or } n=\frac{mp}{100};$$

then $$P\left(1+\frac{p}{100n}\right)^{nt}=P\left(1+\frac{1}{m}\right)^{\frac{mpt}{100}}=P\left\{\left(1+\frac{1}{m}\right)^{m}\right\}^{\frac{pt}{100}}.$$

Suppose now that n, and therefore m, is very large; the interest is thus added on at very short intervals. But when m is very large the expression

$$\left(1+\frac{1}{m}\right)^{m}$$

is *finite*; in fact, it can be shown (see below) that when m becomes infinite that expression converges to the number 2·71828 ..., usually denoted by e. Again, when n becomes infinite the growth of the principal may be said to be *continuous*; in each interval of time, however short, interest is earned and immediately begins to bear interest. The amount A at the end of t years, when interest is added on continuously to the principal, is thus

$$A=Pe^{\frac{pt}{100}}, \text{ since } \underset{m=\infty}{\mathrm{L}}\left(1+\frac{1}{m}\right)^{m}=e.$$

A quantity A which varies with t according to the law

$$A=Pe^{at}$$

is such that, when t increases by equal amounts, A is *multiplied* by equal amounts. Thus, when t increases by h, A will change from

$$Pe^{at} \text{ to } Pe^{a(t+h)}, \text{ or from } Pe^{at} \text{ to } Pe^{at}\times e^{ah};$$

that is, when t increases by h, A is multiplied by e^{ah}.

The function e^t is called **the exponential function,** and is of great importance in mechanics and physics. A simple instance is the law of the density of the air at different heights; as we descend a hill, the density of the air is equally multiplied in equal distances of descent; for the increase in density per foot of descent is due to the weight of a layer which is itself proportional to the density.

We will assume the result

$$\underset{m=\infty}{\mathrm{L}}\left(1+\frac{1}{m}\right)^{m}=e=2\cdot7182818\ldots\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

and refer for the proof to the author's *Calculus*, §§ 48, 49. We give the following as a *suggestion*, not as a proof. Expanding by the binomial theorem, we have

$$\left(1+\frac{1}{m}\right)^m = 1 + m \cdot \frac{1}{m} + \frac{m(m-1)}{1\,.\,2} \cdot \frac{1}{m^2} + \frac{m(m-1)(m-2)}{1\,.\,2\,.\,3} \cdot \frac{1}{m^3} + \ldots$$

$$= 1 + \frac{1}{1} + \frac{1-\frac{1}{m}}{1\,.\,2} + \frac{\left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)}{1\,.\,2\,.\,3} + \ldots,$$

the $(r+1)^{\text{th}}$ term being

$$\frac{\left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)\left(1-\frac{3}{m}\right)\ldots\left(1-\frac{r-1}{m}\right)}{1\,.\,2\,.\,3\ldots(r-1)r}.$$

When m becomes infinite, each of the fractions

$$1-\frac{1}{m},\quad 1-\frac{2}{m},\quad 1-\frac{3}{m},\quad \ldots$$

becomes 1, and we find

$$\underset{m=\infty}{\mathrm{L}}\left(1+\frac{1}{m}\right)^m = 1 + \frac{1}{1} + \frac{1}{1\,.\,2} + \frac{1}{1\,.\,2\,.\,3} + \frac{1}{1\,.\,2\,.\,3\,.\,4} + \ldots, \quad \ldots\ldots\ldots(2)$$

the $(r+1)^{\text{th}}$ term being $1/r!$.

The series (2) is an infinite series, but a few terms, say the first 10, give a very good approximation to its value. The error due to neglecting all the terms after the $(r+1)^{\text{th}}$ is less than $1/r(r!)$.

For purposes of reference, the expression of e^x as a power series may also be given.

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + R_n \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

where R_n is a quantity, depending on x and n, which is small when n is large, and which converges to zero when n becomes infinite. If x_1 is the numerical value of x (that is, x itself when x is positive, but $-x$ when x is negative) R_n is less than

$$x_1^{n+1} \div (n+1-x_1)(n!).$$

Equation (3) holds for every value of x; when x is fairly small a few terms yield a fair approximation to the value of e^x.

61. Derivatives and Integrals of $\log x$ and e^x. We do not, to begin with, specify the base of the logarithms.

$$D_x \log x = \underset{\delta x=0}{\mathrm{L}} \frac{\log(x+\delta x) - \log x}{\delta x} = \underset{\delta x=0}{\mathrm{L}} \frac{1}{\delta x} \log\left(1+\frac{\delta x}{x}\right).$$

Let $\dfrac{\delta x}{x}=\dfrac{1}{m}$; then, when δx becomes zero, m becomes infinite. Now

$$\frac{1}{\delta x}\log\left(1+\frac{\delta x}{x}\right)=\frac{m}{x}\log\left(1+\frac{1}{m}\right)=\frac{1}{x}\log\left\{\left(1+\frac{1}{m}\right)^m\right\}.$$

But, when m becomes infinite, $(1+1/m)^m$ becomes e; therefore

$$D_x\log x=\frac{1}{x}\log e.$$

If e is the base of the logarithms, then $\log e=1$, and therefore

I. $$\mathbf{D_x\log x=\frac{1}{x}}.$$

Note. Unless the contrary is stated, the base we assume for the logarithm will always be e; we thus avoid the cumbrous factor $\log e$. Logarithms to the base e are called *Napierian*, or *hyperbolic*, or *natural* logarithms so as to distinguish them from logarithms to the base 10, which are called *common* or *Briggian* logarithms. Of course the derivative of $\log_{10}x$ is (as we have just proved) $\dfrac{1}{x}\log_{10}e$.

We can also, by the substitution $u=ax+b$, prove

II. $$\mathbf{D_x\log(ax+b)=\frac{a}{ax+b}}.$$

For, if $u=ax+b$

$$\frac{d.\log(ax+b)}{dx}=\frac{d.\log u}{du}\frac{du}{dx}=\frac{1}{u}.a=\frac{a}{ax+b}.$$

Next, we find the derivative of e^x. Let $y=e^x$; then $x=\log y$ and

$$\frac{dx}{dy}=\frac{1}{y};\quad \frac{dy}{dx}=y=e^x, \text{ so that}$$

III. $$\mathbf{D_x e^x=e^x}.$$

By the substitution $u=ax$, we find

IV. $$\mathbf{D_x e^{ax}=ae^{ax}}.$$

For, $$\frac{d.e^{ax}}{dx}=\frac{d.e^u}{du}\cdot\frac{du}{dx}=e^u.a=ae^{ax}.$$

From these results the following integrals are obtained:

V. $\int \frac{1}{\mathbf{x}} \mathbf{dx} = \log \mathbf{x}$; $\int \frac{1}{\mathbf{ax}+\mathbf{b}} \mathbf{dx} = \frac{1}{\mathbf{a}} \log(\mathbf{ax}+\mathbf{b})$.

VI. $\int \mathbf{e^x dx} = \mathbf{e^x}$; $\int \mathbf{e^{ax} dx} = \frac{1}{\mathbf{a}} \mathbf{e^{ax}}$.

To these we add

VII. $$\int \frac{1}{\sqrt{(\mathbf{x}^2+\mathbf{k})}} \mathbf{dx} = \log \{\mathbf{x} + \sqrt{(\mathbf{x}^2+\mathbf{k})}\}.$$

VIII. $$\int \sqrt{(\mathbf{x}^2+\mathbf{k})} \mathbf{dx} = \tfrac{1}{2}\mathbf{x}\sqrt{(\mathbf{x}^2+\mathbf{k})} + \tfrac{1}{2}\mathbf{k} \log\{\mathbf{x}+\sqrt{(\mathbf{x}^2+\mathbf{k})}\},$$

which are proved in examples 1, 2; k may be either positive or negative, but x^2+k must not, of course, be negative.

Example 1. If $y = \log\{x+\sqrt{(x^2+k)}\}$, find dy/dx.

Let $u = x+\sqrt{(x^2+k)}$, so that $y = \log u$; then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{u}\frac{du}{dx} = \frac{1}{x+\sqrt{(x^2+k)}}\frac{du}{dx}.$$

But $$\frac{du}{dx} = 1 + \tfrac{1}{2}(x^2+k)^{-\frac{1}{2}} \cdot 2x = \frac{x+\sqrt{(x^2+k)}}{\sqrt{(x^2+k)}},$$

so that $$\frac{dy}{dx} = \frac{1}{\sqrt{(x^2+k)}}.$$

From this result we obtain the integral of $1/\sqrt{(x^2+k)}$.

Example 2. If $y = \frac{1}{2}x\sqrt{(x^2+k)} + \frac{1}{2}k\log\{x+\sqrt{(x^2+k)}\}$, find Dy.

Take separately $x\sqrt{(x^2+k)}$ and $k\log\{x+\sqrt{(x^2+k)}\}$. We have

$$D[x\sqrt{(x^2+k)}] = \sqrt{(x^2+k)} + x \cdot \tfrac{1}{2}(x^2+k)^{-\frac{1}{2}} \cdot 2x$$
$$= \frac{2x^2+k}{\sqrt{(x^2+k)}};$$

$$D[k\log\{x+\sqrt{(x^2+k)}\}] = \frac{k}{\sqrt{(x^2+k)}}, \text{ ... (by example 1).}$$

Adding these two results and dividing by 2, we find

$$Dy = \sqrt{(x^2+k)},$$

from which the integral of $\sqrt{(x^2+k)}$ is obtained.

These integrals should be compared with those obtained (§ 58) when the integrand is $1/\sqrt{(a^2-x^2)}$ or $\sqrt{(a^2-x^2)}$. When the coefficient of x^2 is *negative*, the integral contains

inverse circular functions; when the coefficient of x^2 is *positive*, the integral contains logarithms.

Example 3. Differentiate $\log \tan \frac{1}{2}x$.
Let $\tan \frac{1}{2}x = u$; then

$$\frac{d \cdot \log \tan \frac{1}{2}x}{dx} = \frac{d \cdot \log u}{du} \cdot \frac{du}{dx} = \frac{1}{u} \cdot \frac{1}{2} \sec^2 \tfrac{1}{2}x,$$

so that
$$\frac{d \cdot \log \tan \frac{1}{2}x}{dx} = \frac{1}{2 \sin \frac{1}{2}x \cos \frac{1}{2}x} = \frac{1}{\sin x}.$$

From this result we deduce the integrals

$$\int \frac{1}{\sin x} dx = \log \tan \tfrac{1}{2}x, \qquad \int \frac{1}{\cos x} dx = \log \tan\left(\tfrac{1}{2}x + \frac{\pi}{4}\right).$$

To obtain the integral of $1/\cos x$, let $x = u - \pi/2$; the integral is thus reduced to that of $1/\sin u$.

Example 4. Integrate $\log x$.
Apply integration by parts (§ 47); take $\log x$ as the product of 1 and $\log x$, and begin by integrating the factor 1. Thus,

$$\int 1 \cdot \log x dx = \log x \cdot x - \int x \cdot \frac{1}{x} dx = x \log x - \int 1 dx,$$

so that
$$\int \log x dx = x \log x - x.$$

Example 5. Prove that if x is numerically less than 1, or equal to $+1$ (*not* -1),

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \text{ to infinity.}$$

Proceed as in § 59, example 5. Dividing 1 by $1+u$, we find,

$$\frac{1}{1+u} = 1 - u + u^2 - u^3 + \dots \pm u^{n-1} \mp \frac{u^n}{1+u},$$

and therefore, integrating from 0 to x,

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \pm \frac{x^n}{n} \mp \int_0^x \frac{u^n du}{1+u}. \quad \text{.................(i)}$$

Now suppose in the first place that x, and therefore u, is *positive*; we see, exactly as in § 59, that

$$\int_0^x \frac{u^n du}{1+u} < \int_0^x u^n du \text{ or } \frac{x^{n+1}}{n+1}. \quad \text{........................(ii)}$$

If x is not greater than 1 the quantity $x^{n+1}/(n+1)$ is not greater than $1/(n+1)$ and is therefore small when n is large, and converges to zero when n becomes infinite. $\log(1+x)$ is therefore equal to the infinite series; the error, when the series is stopped at the term x^n/n, is less than $x^{n+1}/(n+1)$.

Next suppose x, and therefore u, to be *negative*; let $x = -x_1$, where x_1 is positive and less than 1. Put $-v$ for u in the integral in (i); then

$$\int_0^x \frac{u^n du}{1+u} = -\int_0^{x_1} \frac{(-v)^n dv}{1-v} = -(-1)^n \int_0^{x_1} \frac{v^n dv}{1-v}. \quad \text{.............(iii)}$$

The least value of $1-v$ in the integral is $1-x_1$; therefore, as v increases from 0 to x_1, the integrand $v^n/(1-v)$ is less than $v^n/(1-x_1)$, except for the one value $v = x_1$. Hence as before, by considering areas,

$$\int_0^{x_1} \frac{v^n dv}{1-v} < \int_0^{x_1} \frac{v^n dv}{1-x_1} \text{ or } \frac{x_1^{n+1}}{(1-x_1)(n+1)}. \quad \text{..................(iv)}$$

The limit of $x_1^{n+1}/(1-x_1)(n+1)$ for n becoming infinite is zero, provided x_1 is less than 1, as we have supposed it to be. The series for $\log(1+x)$ therefore holds for negative values of x that are numerically less than 1; the error, when the series is stopped at the term x^n/n, is less than the quantity in (iv).

When $x = 1$ we have the identity,

$$\log 2 = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \ldots, \quad \text{.......................(v)}$$

but this is a very bad series for calculating $\log 2$.

Better series for calculating logarithms may be obtained thus:

Take first the series for $\log(1+x)$,

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots. \quad \text{................(vi)}$$

For x put $-x$; then

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \ldots. \quad \text{............(vii)}$$

Subtract (vii) from (vi); therefore

$$\log\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \ldots\right). \quad \text{..............(viii)}$$

Next let $$\frac{1+x}{1-x} = \frac{y+1}{y},$$

so that $$x = \frac{1}{2y+1} < 1.$$

Equation (viii) now becomes

$$\log(y+1) = \log y + 2\left\{\frac{1}{2y+1} + \frac{1}{3}\left(\frac{1}{2y+1}\right)^3 + \frac{1}{5}\left(\frac{1}{2y+1}\right)^5 + \ldots\right\}, \ldots\text{(ix)}$$

from which $\log(y+1)$ is found when $\log y$ is known. Thus

$$y = 1;\ \log 2 = 2\left\{\frac{1}{3} + \frac{1}{3 \cdot 3^3} + \frac{1}{5 \cdot 3^5} + \frac{1}{7 \cdot 3^7} + \ldots\right\};$$

$$y = 2;\ \log 3 = \log 2 + 2\left\{\frac{1}{5} + \frac{1}{3 \cdot 5^3} + \frac{1}{5 \cdot 5^5} + \frac{1}{7 \cdot 5^7} + \ldots\right\};$$

then $\log 4 = 2 \log 2$; $\log 5$ is obtained by putting 4 for y in (ix), and so on.

It will be a good exercise for the student to calculate the logarithms of the first ten integers; the series (ix) converges very rapidly even when $y=2$.

EXERCISES. XVIII.

1. Differentiate $\log\left(\frac{x+a}{x-a}\right)$ and integrate $\frac{1}{x^2-a^2}$, $(x^2>a^2)$.

2. Differentiate $\log\left(\frac{a+x}{a-x}\right)$ and integrate $\frac{1}{a^2-x^2}$, $(x^2<a^2)$.

Differentiate the functions in examples 3–34:

3. $\log\left(\frac{\sqrt{a}+\sqrt{x}}{\sqrt{a}-\sqrt{x}}\right)$. **4.** $\log\{\sqrt{(x+a)}+\sqrt{(x-a)}\}$.

5. $\log\{x-\frac{1}{2}a+\sqrt{(x^2-ax)}\}$. **6.** $\log\left(\frac{x^2+ax+a^2}{x^2-ax+a^2}\right)$.

7. $\log(\sin x)$. **8.** $\log(\cos x)$. **9.** $\log\left(\frac{1+\sin x}{1-\sin x}\right)$.

10. $\log\left(\frac{1+\cos x}{1-\cos x}\right)$. **11.** $\log(1+\sin x)$. **12.** $\log(x+\sin x)$.

13. $x+\log(\sin x+\cos x)$. **14.** $3x-\log(\sin x+2\cos x)$.

15. $x\tan^{-1}x-\log\sqrt{(1+x^2)}$. **16.** $x+\log(x^2-x+1)+\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)$.

17. $\frac{1}{2}(x+1)\sqrt{(x^2+2x+3)}+\log\{x+1+\sqrt{(x^2+2x+3)}\}$.

18. $x\log x$. **19.** $x^n\log x$. **20.** $\frac{1}{2x^2}+\frac{1}{x}+\log\left(\frac{x-1}{x}\right)$.

21. xe^x. **22.** xe^{-x}. **23.** x^ne^x. **24.** x^ne^{-x}. **25.** 10^x.

26. 10^{-x}. **27.** $e^x\sin x$. **28.** $e^x\cos x$. **29.** $e^{-3x}\sin(4x+5)$.

30. $e^{-3x}\cos(4x+5)$. **31.** $e^{ax}\sin(bx+c)$. **32.** $e^{ax}\cos(bx+c)$.

33. $\frac{a}{2}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)$. **34.** $\frac{a}{2}\left(e^{\frac{x}{a}}-e^{-\frac{x}{a}}\right)$.

35. Show that the operation of differentiating e^{ax} is equivalent to multiplying e^{ax} by a, and find the n^{th} derivative of e^{ax}.

36. Find the 2nd, 3rd, and n^{th} derivatives of $\log x$.

37. OM is the abscissa and MP the ordinate at the point $P(\alpha, \beta)$ on the hyperbola $x^2/a^2-y^2/b^2=1$, α and β being both positive. Find the area AMP, A being the vertex nearest P; and show that the expression for the area may be put in the form

$$\tfrac{1}{2}\alpha\beta-\tfrac{1}{2}ab\log\left(\frac{\alpha}{a}+\frac{\beta}{b}\right).$$

38. Show that, if the area of the *sector OAP* (example 37, O being the origin of coordinates) is denoted by $\frac{1}{2}u$,

$$u=ab\log\left(\frac{\alpha}{a}+\frac{\beta}{b}\right).$$

If $a=1$, $b=1$, show that

$$\alpha=\tfrac{1}{2}(e^{u}+e^{-u}),\quad \beta=\tfrac{1}{2}(e^{u}-e^{-u}).$$

39. The curve given by the equation $y=\frac{a}{2}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)$ is called a **catenary**; graph the curve (§ 68, Fig. 36).

If s is the arc, measured from the vertex $(0, a)$, prove that

$$\text{(i)}\ \frac{ds}{dx}=\tfrac{1}{2}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right),\quad \text{(ii)}\ s=\frac{a}{2}\left(e^{\frac{x}{a}}-e^{-\frac{x}{a}}\right),\quad \text{(iii)}\ y^2=s^2+a^2.$$

40. The curve given by the equation $y=a\log\sec(x/a)$ is called **the catenary of uniform strength**; graph the curve.

If s is the arc, measured from the origin $(0, 0)$, show that

$$\text{(i)}\ \frac{ds}{dx}=\sec\frac{x}{a},\quad \text{(ii)}\ s=a\log\tan\left(\frac{x}{2a}+\frac{\pi}{4}\right).$$

41. Show that the arc of the parabola $y=x^2/p$, measured from the vertex up to the point whose abscissa is x, is given by the equation

$$s=\frac{x}{2p}\sqrt{(4x^2+p^2)}+\frac{p}{4}\log\left\{\frac{2x+\sqrt{(4x^2+p^2)}}{p}\right\}.$$

$\Big[\frac{ds}{dx}=\sqrt{\left(1+\frac{4x^2}{p^2}\right)}=\frac{2}{p}\sqrt{\left(x^2+\frac{p^2}{4}\right)}.$

Also, $\int\sqrt{\left(x^2+\frac{p^2}{4}\right)}dx=\frac{1}{2}x\sqrt{\left(x^2+\frac{p^2}{4}\right)}+\frac{1}{2}\cdot\frac{p^2}{4}\log\left\{x+\sqrt{\left(x^2+\frac{p^2}{4}\right)}\right\}.$

See, for an approximate formula, Exercises IX., 16.$\Big]$

Find the integrals of the functions in examples 42–48; compare § 59.

42. $\frac{1}{\sqrt{(x^2-1)}}$. **43.** $\frac{1}{\sqrt{(3x^2+1)}}$. **44.** $\frac{1}{\sqrt{(x^2+2x+2)}}$.

45. $\frac{1}{\sqrt{(3x^2+4x-5)}}$. **46.** $\frac{1}{\sqrt{(x^2+2ax+b^2)}}$.

47. $\sqrt{\{(x-1)(x-2)\}}$. **48.** $\sqrt{(x^2+2ax+b^2)}$.

Apply the method of integration by parts to integrate the functions in examples 49-57.

49. $x\log x$. **50.** $x^2\log x$. **51.** $x^n\log x$. **52.** xe^x. **53.** x^2e^x.

54. x^3e^x. **55.** xe^{-x}. **56.** x^2e^{-x}. **57.** x^3e^{-x}.

58. A rope, which is tightly stretched, makes a half turn round a cylindrical post, the part APB of the rope in contact with the post lying in a plane which intersects the axis of the post at O. The angle AOP is θ radians and the tension at P is T. Assuming that the rate $-dT/d\theta$ at which the tension decreases is equal to μT, where μ is the coefficient of friction between the rope and the post, and that the tensions at A and B are respectively T_1 and T_2 $(T_1 > T_2)$, find the relation between T_1 and T_2.

Find also the relation when the rope makes (i) one complete turn, (ii) two complete turns round the post.

59. Calculate, by equation (ix), example 5, § 61, to 5 decimal places the Napierian logarithms of 2, 3, 4, 5, 6, 7, 8, 9, 10, and then find the common logarithms to 4 decimal places.

60. If $u = \tan\frac{1}{2}x$, show that

$$\int\frac{dx}{3+5\cos x} = \int\frac{du}{4-u^2} = \frac{1}{4}\log\left(\frac{2+\tan\frac{1}{2}x}{2-\tan\frac{1}{2}x}\right).$$

Prove also that, if $a^2 < b^2$, and a positive,

$$\int\frac{dx}{a+b\cos x} = \frac{1}{\sqrt{(b^2-a^2)}}\log\left(\frac{\sqrt{(b+a)}+u\sqrt{(b-a)}}{\sqrt{(b+a)}-u\sqrt{(b-a)}}\right)$$

where $u = \tan\frac{1}{2}x$. (Compare Exercises XVII., 37–40.)

62. Important Examples.

The following examples are of great importance in all applications of the exponential function.

Example 1. Find the integrals of $e^{ax}\cos(bx+c)$ and $e^{ax}\sin(bx+c)$.

Note that these functions differ only in this, that one has the cosine and the other the sine; the angle and the exponential are the same.

The integrals may be deduced very readily from the derivatives of the functions. Let

$$u = e^{ax}\cos(bx+c), \quad v = e^{ax}\sin(bx+c).$$

Differentiate, arranging so that in the derivatives the cosine term comes first; we find

$$Du = ae^{ax}\cos(bx+c) - be^{ax}\sin(bx+c) = au - bv,$$
$$Dv = be^{ax}\cos(bx+c) + ae^{ax}\sin(bx+c) = bu + av.$$

Now solve for u and v, in terms of Du and Dv; we obtain

$$u = \frac{aDu + bDv}{a^2+b^2}, \quad v = \frac{aDv - bDu}{a^2+b^2}.$$

But the integrals of Du and Dv are u and v respectively; we therefore find, by integrating the last two equations and putting for u and v their values,

$$\int e^{ax}\cos(bx+c)\,dx = \frac{e^{ax}\{a\cos(bx+c) + b\sin(bx+c)\}}{a^2+b^2}, \quad \text{.........} (c)$$

$$\int e^{ax}\sin(bx+c)\,dx = \frac{e^{ax}\{a\sin(bx+c) - b\cos(bx+c)\}}{a^2+b^2}. \quad \text{.........} (s)$$

If the structure of these integrals be examined it will be easy to deduce a rule for remembering them.

The more usual method of evaluating the integrals is that of integration by parts. Let

$$P=\int e^{ax}\cos(bx+c)dx, \quad Q=\int e^{ax}\sin(bx+c)dx.$$

To evaluate P, begin by integrating e^{ax}; we find

$$P=\frac{e^{ax}}{a}\cos(bx+c)-\int\frac{e^{ax}}{a}\{-b\sin(bx+c)\}dx=\frac{e^{ax}\cos(bx+c)}{a}+\frac{bQ}{a},$$

so that $$aP-bQ=e^{ax}\cos(bx+c).$$

Integrating $e^{ax}\sin(bx+c)$ in a similar manner, we find

$$bP+aQ=e^{ax}\sin(bx+c).$$

These two equations, when solved for P and Q, give the same values for the integrals as before.

Example 2. Evaluate $\int_0^N e^{-kx}\sin bx dx$, k and N being positive.

In (s), example 1, put $-k$ for a and 0 for c; then

$$\int_0^N e^{-kx}\sin bx dx=\left[\frac{e^{-kx}(-k\sin bx-b\cos bx)}{k^2+b^2}\right]_0^N$$

$$=\frac{-e^{-kN}(k\sin bN+b\cos bN)}{k^2+b^2}+\frac{b}{k^2+b^2}. \text{ ...... (i)}$$

If we define the integral when one of its limits is infinite as follows

$$\int_0^\infty e^{-kx}\sin bx dx=\underset{N=\infty}{\mathrm{L}}\int_0^N e^{-kx}\sin bx dx, \text{ .... (ii)}$$

we can readily find its value in this case. The limit of e^{-kN} is zero when N becomes infinite; also the functions $\sin bN$ and $\cos bN$ can not be numerically greater than unity. Therefore the first fraction in (i) converges to zero when N becomes infinite, and we find

$$\int_0^\infty e^{-kx}\sin bx dx=\frac{b}{k^2+b^2}.$$

We proceed in a similar way in all cases in which one of the limits of an integral is infinite. For example

$$\int_0^\infty e^{-x}dx=\underset{N=\infty}{\mathrm{L}}\int_0^N e^{-x}dx=\underset{N=\infty}{\mathrm{L}}(1-e^{-N})=1,$$

$$\int_0^\infty\frac{dx}{1+x^2}=\underset{N=\infty}{\mathrm{L}}\int_0^N\frac{dx}{1+x^2}=\underset{N=\infty}{\mathrm{L}}\tan^{-1}N=\frac{\pi}{2}.$$

Example 3. Find the turning values of $e^{-ax}\sin(bx+c)$, a and b being positive.

Let $e^{-ax}\sin(bx+c)=f(x)$; then

$$f'(x)=-ae^{-ax}\sin(bx+c)+be^{-ax}\cos(bx+c)$$
$$=-e^{-ax}\{a\sin(bx+c)-b\cos(bx+c)\}. \text{ ....(1)}$$

This expression may be put in a more convenient form. Choose R and θ so that $R\cos\theta=a$, $R\sin\theta=b$; these equations give

$$R=\surd(a^2+b^2),\quad \tan\theta=b/a. \qquad (2)$$

Since a and b are positive, the smallest value of θ is an acute angle; this is the value to be chosen, which should be expressed in radians. Equation (1) now becomes

$$f'(x)=-Re^{-ax}\{\cos\theta\sin(bx+c)-\sin\theta\cos(bx+c)\}$$
$$=-Re^{-ax}\sin(bx+c-\theta). \qquad (3)$$

From (3) we see that, to obtain the derivative of $e^{-ax}\sin(bx+c)$, we have simply to multiply $e^{-ax}\sin(bx+c)$ by $-R$ and subtract θ from the angle; both R and θ are independent of c. Hence to obtain the derivative of the function in (3) we multiply it by $-R$ and subtract θ from the angle; thus

$$f''(x)=R^2e^{-ax}\sin(bx+c-2\theta), \qquad (4)$$

as may be easily verified by direct differentiation.

For the turning values we put $f'(x)$ equal to zero; therefore by (3) the turning values are given by

$$\sin(bx+c-\theta)=0,\quad \text{or}\quad bx+c-\theta=n\pi$$

where n is 0, ± 1, ± 2, ... We shall confine our attention to *positive* values of x; therefore, denoting by x_0, x_1, x_2, ... x_n, the values of x for the values 0, 1, 2, ... n of n, we find

$$x_0=\frac{\theta-c}{b},\quad x_1=\frac{\pi+\theta-c}{b},\quad x_2=\frac{2\pi+\theta-c}{b},$$

and, generally,
$$x_n=\frac{n\pi+\theta-c}{b}.$$

It may happen that the values of θ and c are such as to make one or two of the smaller values of x negative.

In particular cases it is easy to decide which are the maximum and which the minimum values of $f(x)$; the general case is easily decided. For
$$f''(x_n)=R^2e^{-ax_n}\sin(bx_n+c-2\theta)=R^2e^{-ax_n}\sin(n\pi-\theta),$$
and therefore, since $R^2e^{-ax_n}\sin\theta$ is positive,

$$f''(x_n)=-,\ \text{if } n=0,\ 2,\ 4,\ 6,\ \dots\ \text{(even integer)}$$
$$=+,\ \text{if } n=1,\ 3,\ 5,\ 7,\ \dots\ \text{(odd integer)}.$$

Hence $f(x)$ is a maximum for the values x_0, x_2, x_4, ... of x, and a minimum for the values x_1, x_3, x_5, ... (§ 24, example 2).

The student should work out, from the beginning, the particular case in which $a=0{\cdot}1$, $b=1$, $c=0$; the graph is shown in Fig. 31. In this case $\tan\theta=10$ and θ is 84° 18′ or 1·471 radians.

$$x_0=1{\cdot}47,\quad x_1=4{\cdot}61,\quad x_2=7{\cdot}75,\quad x_3=10{\cdot}90,$$
$$f(x_0)=0{\cdot}86,\ f(x_1)=-0{\cdot}63,\ f(x_2)=0{\cdot}46,\ f(x_3)=-0{\cdot}33.$$

It will be a good exercise for the student to show that the ordinate and the gradient of $e^{-ax}\sin(bx+c)$ are equal respectively to the ordinate and the gradient of e^{-ax}, for those values of x for which

$$bx+c=2m\pi+\tfrac{1}{2}\pi\quad (m=0,\ 1,\ 2,\ \dots).$$

The two graphs therefore touch at the points whose abscissae are given by these values of x.

Since $\cos(bx+c)=\sin(bx+c+\frac{1}{2}\pi)$, the turning values of $e^{-ax}\cos(bx+c)$ may be obtained from those of $e^{-ax}\sin(bx+c)$ by writing $c+\frac{1}{2}\pi$ for c.

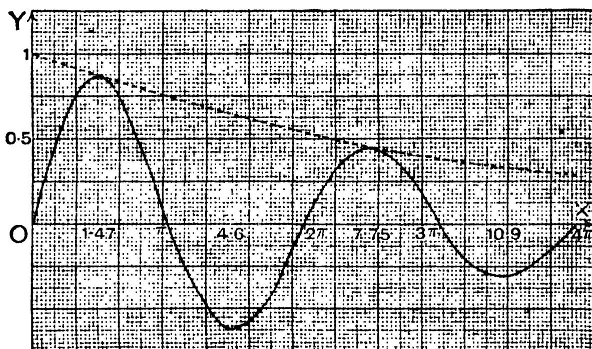


Fig. 31.

The differential equation in the next example occurs very frequently in physical applications; a special case of it has already been discussed in § 43, example 5.

Example 4. Verify that, if $\mu-\frac{1}{4}k^2$ is *positive*, the equation

$$\frac{d^2x}{dt^2}+k\frac{dx}{dt}+\mu x=0 \quad \text{.............................(i)}$$

is satisfied by $x=Ae^{-\frac{1}{2}kt}\cos(nt+B)$, where A, B are any constants and n is given by the equation $n^2=\mu-\frac{1}{4}k^2$.

To verify, we have simply to find $\frac{dx}{dt}$, $\frac{d^2x}{dt^2}$ and to eliminate the two constants A, B from the three equations which give x and the two derivatives. From the equation

$$x=Ae^{-\frac{1}{2}kt}\cos(nt+B) \quad \text{.............................(ii)}$$

we obtain by differentiation

$$\frac{dx}{dt}=-\tfrac{1}{2}kAe^{-\frac{1}{2}kt}\cos(nt+B)-nAe^{-\frac{1}{2}kt}\sin(nt+B)$$

$$=-\tfrac{1}{2}kx-nAe^{-\frac{1}{2}kt}\sin(nt+B). \quad \text{.........................(iii)}$$

Differentiate again; therefore

$$\frac{d^2x}{dt^2}=-\tfrac{1}{2}k\frac{dx}{dt}-n^2Ae^{-\frac{1}{2}kt}\cos(nt+B)+\tfrac{1}{2}knAe^{-\frac{1}{2}kt}\sin(nt+B)$$

$$=-\tfrac{1}{2}k\frac{dx}{dt}-n^2x-\tfrac{1}{2}k\left(\frac{dx}{dt}+\tfrac{1}{2}kx\right) \text{ by (ii) and (iii).}$$

Hence, taking all the terms to the left side, we find

$$\frac{d^2x}{dt^2}+k\frac{dx}{dt}+(n^2+\tfrac{1}{4}k^2)x=0,$$

which is the same as equation (i) if $\mu=n^2+\frac{1}{4}k^2$, or $n^2=\mu-\frac{1}{4}k^2$.

Equation (ii) represents (when k is positive) what is often termed a *damped vibration*; it is a simple harmonic motion with a decreasing amplitude. The amplitude, when t has any value t_1, is $Ae^{-\frac{1}{2}kt_1}$; when t has increased by $\frac{1}{2}T$ (where T is the period $2\pi/n$ of the circular function) the amplitude has decreased to $Ae^{-\frac{1}{2}k(t_1+\frac{1}{2}T)}$. The ratio of the first to the second of these amplitudes is

$$Ae^{-\frac{1}{2}kt_1} : Ae^{-\frac{1}{2}k(t_1+\frac{1}{2}T)} \text{ or } e^{\frac{1}{4}kT};$$

the Napierian logarithm of this ratio, namely $\frac{1}{4}kT$, is called the **logarithmic decrement** of the amplitude.

EXERCISES. XIX.

Calculate the values of the integrals in examples 1–18; k is positive.

1. $\int e^x \sin x\,dx$. **2.** $\int e^x \cos x\,dx$. **3.** $\int e^{-x}\sin x\,dx$.

4. $\int e^{-x}\cos x\,dx$. **5.** $\int e^{-3x}\sin 4x\,dx$. **6.** $\int e^{-4x}\cos 3x\,dx$.

7. $\int e^x\sin^2 x\,dx$. **8.** $\int e^{-x}\cos^2 x\,dx$. **9.** $\int_0^\infty e^{-x}\cos 2x\,dx$.

10. $\int_0^\infty e^{-3x}\cos(4x+5)\,dx$. **11.** $\int_0^\infty e^{-kt}\sin nt\,dt$.

12. $\int_0^\infty e^{-kt}\cos nt\,dt$. **13.** $\int_0^\infty e^{-kt}\sin(nt+a)\,dt$.

14. $\int_0^\infty \frac{dx}{x^2+4}$. **15.** $\int_{-\infty}^\infty \frac{dx}{x^2+4}$. **16.** $\int_0^\infty xe^{-x}\,dx$.

17. $\int_1^\infty \frac{dx}{x^2}$. **18.** $\int_0^\infty \frac{dx}{(2x+1)^3}$.

19. Evaluate $\int_0^\infty \frac{dx}{(x^2+a^2)^2}$, by the substitution $x=a\tan u$.

20. Evaluate

$$\text{(i)} \int_0^\infty \frac{dx}{x^2+2x+2}, \quad \text{(ii)} \int_{-\infty}^\infty \frac{dx}{x^2+2x+2}.$$

21. Prove $$\int_0^{\frac{\pi}{2}} \frac{dx}{a^2\cos^2 x+b^2\sin^2 x}=\int_0^\infty \frac{du}{a^2+b^2u^2}$$
where $u=\tan x$, and evaluate the integral.

Graph the functions in examples 22–25.

22. $e^{-x}\sin 2x$. **23.** $e^{-x}\cos 2x$. **24.** $e^{-10x}\sin 300x$. **25.** $e^{-10x}\sin 500x$.

26. Show that each of the functions given by the following equations satisfies the differential equation of example 4, § 62, k, μ, n having the same meanings as in that example:

(i) $x = Ce^{-\frac{1}{2}kt}\sin(nt+D)$, (ii) $x = e^{-\frac{1}{2}kt}(E\cos nt + F\sin nt)$.

27. Find the value of x which satisfies the differential equation of example 4, § 62, and also the conditions that $x=a$ and $dx/dt=V$ when $t=0$.

28. Verify that, if $\mu - \frac{1}{4}k^2$ is *negative*, the equation

$$\frac{d^2x}{dt^2} + k\frac{dx}{dt} + \mu x = 0$$

is satisfied by $x = e^{-\frac{1}{2}kt}(Ae^{mt} + Be^{-mt})$, where A, B are any constants and m is given by the equation $m^2 = \frac{1}{4}k^2 - \mu$.

29. Verify that the equation

$$\frac{d^2x}{dt^2} - n^2x = 0$$

is satisfied by $x = Ae^{nt} + Be^{-nt}$, where A and B are any constants.

CHAPTER XIII.

APPLICATIONS. CURVATURE. BENDING OF BEAMS. CATENARY. ALTERNATE CURRENTS. DOUBLE INTEGRALS.

63. Geometrical Applications. Let OM be the abscissa and MP the ordinate of the point P (x, y) on the curve whose equation is $y = f(x)$, and let the tangent at P meet the axes at L, K (Fig. 32).

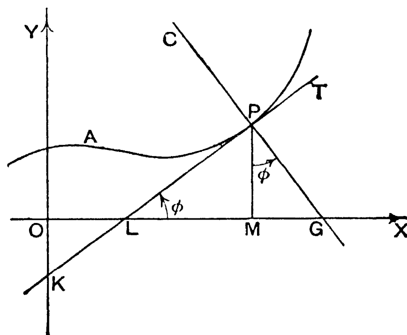


Fig. 32.

The line CPG drawn through P perpendicular to the tangent LP is called the **normal** at P.

When the **tangent** and the **normal** are spoken of as **finite segments**, the portions LP and PG intercepted between P and the x-axis are the segments referred to. In the same way the projections of these segments on the x-axis, namely LM and MG, are called the **subtangent** and the **subnormal** respectively.

These segments may be expressed in terms of the values of x, y, y' at P, where y' means $D_x y$. If the angle XLP is denoted by ϕ, then $y' = \tan\phi$.

$$\begin{aligned}
&\text{Subtangent} = LM = y/\tan\phi = y/y',\\
&\text{Subnormal} \;= MG = y\tan\phi = yy',\\
&\text{Tangent} \quad\;\, = LP = y/\sin\phi = y\sqrt{(1+y'^2)}/y',\\
&\text{Normal} \quad\;\; = PG = y/\cos\phi = y\sqrt{(1+y'^2)},\\
&\quad OL = OM - LM = x - y/y' = (xy' - y)/y',\\
&\quad OK = -OL\tan\phi = -OL\,.\,y' = y - xy'.
\end{aligned}$$

These expressions are true for all positions of P provided the *signs* of the segments be attended to. Thus, if the value given by the formula for LM is *negative*, L will be to the *right* of M; because in Fig. 32, which may be taken as the standard diagram, x, y, y' are all positive, and when y and y' are positive L is to the left of M.

To find the *equations* of the tangent and normal (the unlimited lines) take x, y as current coordinates and denote the values of x, y, y' for the point P by x_1, y_1, y_1'. Then the equations are

$$\text{for the tangent,}\quad y - y_1 = y_1'(x - x_1);$$

$$\text{for the normal,}\quad y - y_1 = -\frac{1}{y_1'}(x - x_1).$$

Note that the gradient of the tangent is y_1' and therefore the gradient of the normal is $-1/y_1'$.

Example 1. Find the subtangent and the subnormal of the ellipse $x^2/a^2 + y^2/b^2 = 1$.

Suppose x and y to be positive; then

$$y = \frac{b}{a}\sqrt{(a^2 - x^2)}\;;\; y' = -\frac{b}{a}\frac{x}{\sqrt{(a^2 - x^2)}},$$

$$\text{subtangent} = LM = \frac{y}{y'} = -\frac{a^2 - x^2}{x},$$

$$\text{subnormal} = MG = yy' = -\frac{b^2}{a^2}x.$$

L is to the right and G to the left of M.

$$OL = OM - LM = x + \frac{a^2 - x^2}{x} = \frac{a^2}{x};$$

therefore $OL\,.\,OM = a^2$.

Example 2. For the hyperbola $xy=c^2$ find $KP:LP$.

$$\frac{KP}{LP}=\frac{OM}{LM}=\frac{xy'}{y}=-\frac{c^2}{x}\div\frac{c^2}{x}=-1.$$

Since the ratio is *negative* P lies between K and L; $KP=PL$. Hence KL is *bisected* at P.

64. Curvature. Let P and Q be two points on a plane curve, ϕ and $\phi+\delta\phi$ the angles which the tangents at P and Q make with the x-axis, s the arc measured from some fixed point on the curve up to P, and δs the arc PQ (Fig. 33).

Definitions. The angle $\delta\phi$ is called the **total curvature** of the arc PQ; the quotient $\delta\phi/\delta s$ is called the **average curvature** of the arc PQ; and the limit of $\delta\phi/\delta s$ when Q approaches P as its limiting position, that is, $d\phi/ds$ is called **the curvature** of the curve at P.

For a circle of radius R the arc δs is equal to $R\delta\phi$, and therefore

$$\frac{\delta\phi}{\delta s}=\frac{1}{R},\quad \frac{d\phi}{ds}=\frac{1}{R};$$

that is, the average curvature of any arc of a circle is equal to the curvature at any point of the circle. In other words, a circle is a curve of constant curvature, and its curvature is equal to the reciprocal of its radius.

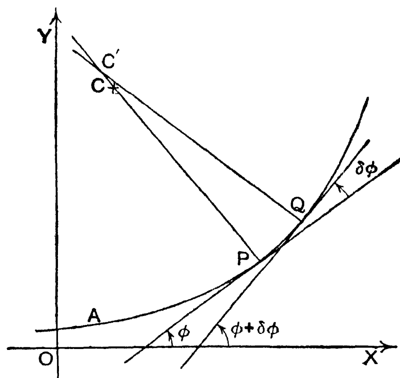


Fig. 33.

65. Circle, Radius, and Centre of Curvature. Let the normals at P and Q (Fig. 33) intersect at C'. We shall

prove that, when Q tends to P as its limiting position, C' will tend to a point C on the normal at P as its limiting position such that PC is equal to $ds/d\phi$.

To prove this, observe that $\angle PC'Q = \delta\phi$ and, from the triangle $PC'Q$,

$$\frac{PC'}{\sin PQC'} = \frac{\text{chord } PQ}{\sin PC'Q} = \frac{\text{chord } PQ}{\text{arc } PQ} \times \frac{\delta s}{\delta\phi} \times \frac{\delta\phi}{\sin \delta\phi}.$$

Now, the limit of $\angle PQC'$ as Q tends to P is 90°; also, the limits of the three fractions last written are 1, $ds/d\phi$, 1 respectively. Therefore, since $\sin 90° = 1$, the limit of PC' is $ds/d\phi$. In other words, C' tends to a point, C say, on PC' such that

$$PC = \frac{ds}{d\phi}, \text{ or } \frac{1}{PC} = \frac{d\phi}{ds}.$$

The circle whose centre is C and radius PC has therefore the same tangent and the same curvature as the curve has at P. This circle is called **the circle of curvature**, its radius PC (which is equal to $ds/d\phi$) **the radius of curvature**, and its centre C **the centre of curvature** at the point P.

We shall generally use R for the radius of curvature; its reciprocal $1/R$ will therefore be the curvature.

66. Expression for the Curvature. The curvature may be expressed in terms of the first and second derivatives of the ordinate at the point. For, since

$$\tan\phi = \frac{dy}{dx}, \quad \cos\phi = \frac{dx}{ds} = \frac{1}{\sec\phi},$$

we get by differentiating the first equation with respect to s

$$\frac{d\tan\phi}{d\phi} \times \frac{d\phi}{ds} = \frac{d}{dx}\left(\frac{dy}{dx}\right) \times \frac{dx}{ds},$$

that is
$$\sec^2\phi \times \frac{d\phi}{ds} = \frac{d^2y}{dx^2} \div \sec\phi,$$

and therefore
$$\frac{d\phi}{ds} = \frac{d^2y}{dx^2} \div \sec^3\phi.$$

But
$$\sec^2\phi = 1 + \tan^2\phi = 1 + \left(\frac{dy}{dx}\right)^2,$$

and therefore $$\frac{1}{R}=\frac{d\phi}{ds}=\frac{d^2y}{dx^2}\div\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}, \quad\text{.............. (1)}$$

so that $$R=\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}\div\frac{d^2y}{dx^2}. \quad\text{.................... (2)}$$

If the gradient dy/dx is so small that for all values of x within the range considered its square may be rejected, equation (1) becomes

$$\frac{1}{R}=\frac{d\phi}{ds}=\frac{d^2y}{dx^2}. \quad\text{......................... (3)}$$

The value for the curvature given by (3) is that generally used in the approximate theory of the bending of beams.

Equation (3) may also be proved independently. For, if the gradient is very small, ϕ is approximately equal to $\tan\phi$ (see § 41, example 2) and δs to its projection δx. Hence, approximately,

$$\frac{\delta\phi}{\delta s}=\frac{\delta.\tan\phi}{\delta x}=\frac{\delta.\frac{dy}{dx}}{\delta x}\text{; therefore, }\frac{d\phi}{ds}=\frac{d^2y}{dx^2}.$$

Example 1. The parabola $y=x^2/p$.

$$\frac{dy}{dx}=\frac{2x}{p},\quad\frac{d^2y}{dx^2}=\frac{2}{p};$$

$$\frac{1}{R}=\frac{d\phi}{ds}=\frac{d^2y}{dx^2}\div\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}=\frac{2}{p}\div\left\{1+\frac{4x^2}{p^2}\right\}^{\frac{3}{2}};$$

$$\frac{1}{R}=\frac{2p^2}{(p^2+4x^2)^{\frac{3}{2}}};\quad R=\frac{(p^2+4x^2)^{\frac{3}{2}}}{2p^2}.$$

Near the origin x^2 is very small; hence near the origin the curvature is simply $2/p$.

Example 2. The ellipse $x^2/a^2+y^2/b^2=1$.

$$\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0,\quad\frac{dy}{dx}=-\frac{b^2x}{a^2y},$$

$$\frac{d^2y}{dx^2}=-\frac{a^2yb^2-b^2xa^2\frac{dy}{dx}}{a^4y^2}=-\frac{a^2b^2y+\frac{b^4x^2}{y}}{a^4y^2}.$$

But $b^2x^2+a^2y^2=a^2b^2$; simplifying a little, we find

$$\frac{d^2y}{dx^2}=-\frac{b^4}{a^2y^3};\quad\frac{1}{R}=-\frac{a^4b^4}{(b^4x^2+a^4y^2)^{\frac{3}{2}}}.$$

The negative sign arises from the negative sign of $\frac{d^2y}{dx^2}$ (y being assumed positive); the root implied in the index 3/2 is taken as positive. In the standard figure (Fig. 33) $\frac{d^2y}{dx^2}$ is positive. When $\frac{d^2y}{dx^2}$ is positive the gradient $\frac{dy}{dx}$ is an increasing function and the curve is concave upwards, as it is near P in Fig. 33; but when $\frac{d^2y}{dx^2}$ is negative the gradient is a decreasing function and the curve is convex upwards for such values. (Exercises IV., example 54.)

EXERCISES. XX.

1. Find the equations of the tangent and normal to the circle $x^2+y^2=R^2$ at the point (x_1, y_1) on the circle.

If the tangent cut the x-axis at T, show that $OM \, . \, OT$ is equal to R^2 where $x_1=OM$.

2. Find the equations of the tangent and normal at the point (x_1, y_1) on the parabola $y^2=4ax$. What form do the equations take if $x_1=at^2$, $y_1=2at$?

Show that the subtangent is bisected at the vertex.

3. Show that in the parabola $y^2=4ax$ the subnormal is constant.

Conversely, show that if the subnormal of a curve is constant, equal to $2a$ say, the curve is a parabola $y^2=4ax+C$, where C is any constant.

4. Find the equations of the tangent and normal at the point (x_1, y_1) on the ellipse $x^2/a^2+y^2/b^2=1$. What form do these equations take if $x_1=a\cos\theta$, $y_1=b\sin\theta$?

5. C is the centre of an ellipse, $A'A$ and $B'B$ the major and minor axes; M and m are the projections on the major and minor axes of a point P on the ellipse and T, t are the points where the tangent at P cuts the axes. Prove

$$CM \, . \, CT=CA^2, \quad Cm \, . \, Ct=CB^2.$$

Establish corresponding theorems for the hyperbola.

6. Using the notation of § 63, show that for the adiabatic curves $yx^n=c^{n+1}$

$$KP : LP= -n:1.$$

Explain the meaning of the minus sign.

7. In the semi-cubical parabola $ay^2=x^3$ show that

$$LM=\tfrac{2}{3}x, \quad MG=\frac{3}{2}\frac{x^2}{a}, \quad MG=\frac{27}{8}\frac{LM^2}{a}.$$

8. Show that for the curve $y=ce^{\frac{x}{a}}$ the subtangent is constant.

Conversely, show that if the subtangent of a curve is constant, equal to a say, the curve is $y=Ce^{\frac{x}{a}}$, where C is any constant. (Compare example 3.)

9. Find the subtangent, the subnormal and the normal of the catenary $y=\frac{1}{2}a(e^{\frac{x}{a}}+e^{-\frac{x}{a}})$. Show that the perpendicular from the foot of the ordinate at any point to the tangent at that point is of constant length. (See § 68.)

10. Find the subtangent and subnormal of the curve of sines, $y=a\sin(x/b)$.

11. Find the curvature *at the origin* of the curves

(i) $y=x^2$, (ii) $y=x^2+x^3$, (iii) $y=ax^2$, (iv) $y=ax^2+bx^3+cx^4$.

12. Find the curvature *at the origin* of the curves

(i) $y=x+x^2$, (ii) $y=x+x^2+x^3$;

(iii) $y=ax+bx^2$, (iv) $y=ax+bx^2+cx^3+dx^4$.

13. The radius of curvature at the point (x, y) on the hyperbola $xy=c^2$ is

$$R=(x^2+y^2)^{\frac{3}{2}}/2c^2.$$

14. The radius of curvature of the catenary of example 9 is y^2/a, and that of the catenary of uniform strength $y=c\log\sec(x/c)$ is $c\sec(x/c)$.

15. Show that *at the origin* on the curve

$$y=2x+3x^2-2xy+y^2,$$

the radius of curvature is $\frac{5}{8}\sqrt{5}$.

67. Bending of a Beam. On a heavy uniform beam, which rests horizontally on two supports near its ends, a load is placed and the beam bends slightly. It is required to find an expression for the couple called into play by the stresses acting across any given cross-section which is originally vertical and perpendicular to the length of the beam, the cross-section being assumed to remain plane when the beam is bent.

In the usual theory it is further assumed that there is one set of filaments, running from end to end of the beam and forming what is called the neutral surface, that are neither extended nor contracted. Let $ABDC$ (Fig. 34) be a vertical section parallel to the length of the beam and let EF be the line in which the section cuts the neutral surface. EF is not altered in length but is curved slightly. Filaments in the section $ABDC$ parallel to the original position

of EF are bent and contracted when they lie between EF and CD, but are bent and elongated when they lie between EF and AB.

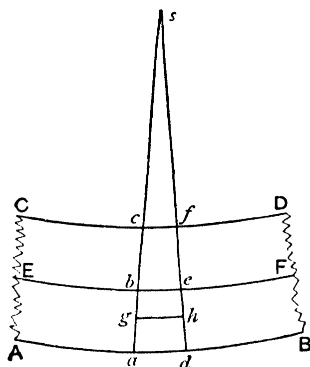


Fig. 34.

To find the stress-couple, consider the given cross-section and another cross-section which is originally parallel and very close to it; both remain plane after bending but are inclined to each other at a small angle. Let abc, def be the lines in which they intersect the section $ABDC$, and let these intersect at s, the angle bse being very small.

The filaments which in the strained position are represented by gh and be were originally equal and parallel; be is unchanged in length but is slightly bent, while gh is bent and is also longer than in the unstrained position. To a sufficient approximation we have

$$\frac{gh}{be}=\frac{sg}{sb}=\frac{sb+bg}{sb}=1+\frac{bg}{sb},$$

and therefore

$$\frac{gh-be}{be}=\frac{bg}{sb}.$$

But $(gh-be)/be$ is the unital extension of the filament gh; the unital extension therefore, if bg be denoted by x, is x/sb. When g lies between b and a the filament is extended and x is positive; when g lies between b and c the filament is contracted and x is negative.

Now, let δA be an element of the cross-section through def, the point h being within δA, and let E be the Young's modulus of the material. Then the stress exerted across δA by the material to the right on the material to the left is

$$\frac{Ex\,\delta A}{sb}.$$

But, as the cross-section through def is taken closer and closer to the cross-section through abc, the line sb becomes

the radius of curvature R at b of the curve be into which the beam is bent. Hence the stress across a small area δA including the point y is $Ex\delta A/R$. By § 39, example 5, we now find for the stress-couple the expression EI/R.

Example 1. A beam of uniform cross-section and length L with a load W at the middle, the weight of the beam being neglected.

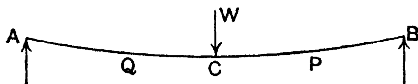


Fig. 35.

When the coordinates of a point of the beam are spoken of, it is to be understood that the point is on the unstretched line EF which represents the curve into which the beam is bent.

Take the origin at the mid-point C of the beam and let the x-axis be horizontal and the y-axis vertical, the positive direction of the y-axis being upwards. Let P be the point (x, y).

The bending moment M at P is the moment about the horizontal line in the section through P of all the applied forces on either side of the section. The only applied force to the right of P is the reaction at B which is equal to $\frac{1}{2}W$; therefore

$$M = \tfrac{1}{2}W(\tfrac{1}{2}L - x). \quad \text{.................................(1)}$$

The bending moment is numerically equal to the stress-couple; since the bending is slight the approximate value for $1/R$ may be used. Hence we have

$$EI\frac{d^2y}{dx^2} = \tfrac{1}{2}W(\tfrac{1}{2}L - x). \quad \text{............................(2)}$$

At the origin C both y and dy/dx are zero, so that the constants of integration are zero. Integrating, we find as the equation of the curve into which the beam is bent

$$EIy = \tfrac{1}{8}Wx^2(L - \tfrac{2}{3}x). \quad \text{............................(3)}$$

The greatest deflection, y_1 say, is given by $x = \frac{1}{2}L$; therefore

$$y_1 = WL^3/48EI. \quad \text{..................................(4)}$$

If the section is a rectangle of depth a and breadth b then $I = \frac{1}{12}ab \times a^2$, and therefore

$$y_1 = WL^3/4Ea^3b. \quad \text{..............................(5)}$$

If the origin is taken at A and if y is measured *downwards* (x being measured from A towards B) then, as x increases, the gradient dy/dx *decreases* so that d^2y/dx^2 is *negative*. The coordinates of Q, a point between A and C, being (x, y) the bending moment at Q is $\frac{1}{2}Wx$. Hence the equation for the curve of the beam is

$$-EI\frac{d^2y}{dx^2} = \tfrac{1}{2}Wx. \quad \text{..................................(6)}$$

Integrating and determining the constants so that $y=0$ when $x=0$, and $dy/dx=0$ when $x=\frac{1}{2}L$ we find

$$EIy=\tfrac{1}{16}Wx(L^2-\tfrac{4}{3}x^2),$$

which of course gives the same maximum deflection as before.

Example 2. A beam of uniform cross-section and length L with a uniformly distributed load w per unit of length.

Take the same coordinates as for equation (6) of the last example. The bending moment at Q is

$$M=\tfrac{1}{2}wL\times x-wx\times\tfrac{1}{2}x=\tfrac{1}{2}w(Lx-x^2). \quad \text{.....(1)}$$

Hence for the curve of the beam we find the equation

$$-EI\frac{d^2y}{dx^2}=\tfrac{1}{2}w(Lx-x^2). \quad \text{.....(2)}$$

Integrating and determining the constants so that $y=0$ when $x=0$ and $x=L$, we obtain

$$EIy=\frac{wx}{24}(L^3-2Lx^2+x^3). \quad \text{.....(3)}$$

The greatest deflection y_1, given by $x=\frac{1}{2}L$, is

$$y_1=\frac{5wL^4}{384EI}. \quad \text{.....(4)}$$

If the beam is not merely supported at A, B but *tangentially fixed*, that is, fixed so that the ends are kept horizontal (for example, by being built in) a couple K is required for this fixation. The span being L and the upward reaction being $\frac{1}{2}wL$ at A, B as before, then the equation for the curve is

$$EI\frac{d^2y}{dx^2}=K-\tfrac{1}{2}w(Lx-x^2). \quad \text{.....(5)}$$

At A both y and dy/dx are zero; therefore

$$EIy=\tfrac{1}{2}Kx^2-\tfrac{1}{24}wx^3(2L-x). \quad \text{.....(6)}$$

Since $y=0$ at the end B where $x=L$, we find, by putting $x=L$ in (6), that $K=\frac{1}{12}wL^2$. Therefore

$$EIy=\tfrac{1}{24}wx^2(L-x)^2. \quad \text{.....(7)}$$

We see from (7) that dy/dx is also zero at B, as it should be.

The greatest deflection, given by $x=\frac{1}{2}L$, is y_2 where

$$y_2=wL^4/384EI,$$

which is one-fifth of the deflection when the ends are simply supported.

68. The Common Catenary. A uniform flexible chain of weight w per unit of length is suspended from two points A and B; to find the curve which the chain assumes.

Any small piece PQ of length δs is in equilibrium under the action of three forces, namely:—its weight $w\delta s$ acting vertically downwards and the tensions, T and $T+\delta T$ say,

acting at the ends P and Q in the direction of the tangents. The three forces are therefore concurrent (Fig. 36).

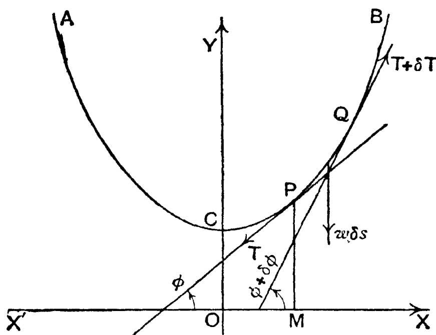


Fig. 36.

Let ϕ and $\phi+\delta\phi$ be the angles which the tangents at P and Q respectively make with the (horizontal) x-axis. First resolve horizontally; therefore

$$(T+\delta T)\cos(\phi+\delta\phi)-T\cos\phi=0, \text{ that is, } \delta(T\cos\phi)=0.$$

Hence, the horizontal component of the tension is constant; if this constant is T_0, we get

$$T\cos\phi=T_0. \quad\text{.........................}(1)$$

Next resolve vertically; therefore

$$(T+\delta T)\sin(\phi+\delta\phi)-T\sin\phi=w\delta s, \text{ that is, } \frac{\delta(T\sin\phi)}{\delta s}=w.$$

Taking the limit for δs converging to zero, we find

$$\frac{d(T\sin\phi)}{ds}=w. \quad\text{.........................}(2)$$

Using (1) and noting that (compare § 66)

$$\frac{d\tan\phi}{ds}=\frac{d\tan\phi}{dx}\cdot\frac{dx}{ds}=\frac{d^2y}{dx^2}\div\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}},$$

we get for equation (2) the form

$$\frac{d^2y}{dx^2}\div\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}=\frac{w}{T_0}, \quad\text{.................}(3)$$

which is the differential equation of the curve; the variables x, y, ϕ here refer to the point P.

To integrate, let dy/dx be denoted by u for the moment; also put a for T_0/w. Equation (3) thus becomes

$$\frac{1}{\sqrt{(1+u^2)}}\frac{du}{dx}=\frac{1}{a}, \text{ or } \frac{du}{\sqrt{(1+u^2)}}=\frac{dx}{a}. \quad \ldots\ldots\ldots\ldots(4)$$

Integrate, choosing the constant of integration so that u may be zero when x is zero; we find

$$\log\{u+\sqrt{(1+u^2)}\}=\frac{x}{a}, \text{ or } e^{\frac{x}{a}}=u+\sqrt{(1+u^2)}. \quad \ldots\ldots(5)$$

Again $e^{-\frac{x}{a}}=1/\{u+\sqrt{(1+u^2)}\}=-u+\sqrt{(1+u^2)}. \quad \ldots\ldots(6)$

Now, from (5) and (6) we get, by subtracting and dividing by 2,

$$\frac{dy}{dx}=u=\tfrac{1}{2}\left(e^{\frac{x}{a}}-e^{-\frac{x}{a}}\right), \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

and therefore

$$y=\frac{a}{2}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right), \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

the constant of integration being chosen so that $y=a$ when $x=0$. Equation (8) is the required equation.

The curve given by equation (8) is called the common catenary, or simply the catenary. Its graph is shown in Fig. 36.

EXERCISES. XXI.

In the following examples the notation of § 67 is adhered to; the beam is supposed to be of uniform cross-section and to be horizontal in the unstrained position.

1. For a beam tangentially fixed at A and loaded at the end B, the weight of the beam being neglected, the curve is (origin at A)

$$EIy=\frac{1}{2}Wx^2(L-\frac{1}{3}x); \quad y_1=\frac{1}{3}\frac{WL^3}{EI}.$$

2. For a beam tangentially fixed at A and free at the end B, with a uniformly distributed load the curve is

$$EIy=\frac{1}{24}wx^2(6L^2-4Lx+x^2); \quad y_1=\frac{1}{8}\frac{wL^4}{EI}.$$

3. For a beam supported at its middle point and uniformly loaded, the curve is (origin at C, the positive direction of the y-axis being *downwards*)

$$EIy=\frac{1}{48}wx^2(3L^2-4Lx+2x^2); \quad y_1=\frac{1}{128}\frac{wL^4}{EI}.$$

4. If in example 1, § 67, the origin is at A and the positive direction of the y-axis is downwards, show that for a point such as P between C and B the equation is

$$EI\frac{d^2y}{dx^2} = -\tfrac{1}{2}W(L-x).$$

Integrate the equation.

5. Show that if, in equation (7), example 2, § 67, the origin is at the middle point of AB that equation becomes

$$EIy = \tfrac{1}{24}w(\tfrac{1}{4}L^2 - x^2)^2,$$

the positive direction of the y-axis being downwards.

6. For a beam supported at its ends and loaded at a point D such that $AD = a$, $DB = b$, the equation for a point between A and D is

$$EI\frac{d^2y}{dx^2} = -\frac{bW}{a+b}x, \qquad \text{(i)}$$

and for a point between B and D is

$$EI\frac{d^2y}{dx^2} = -\frac{aW}{a+b}(a+b-x). \qquad \text{(ii)}$$

The integral of (i) is

$$EIy = \frac{ab(a+2b)W}{6(a+b)}x - \frac{bW}{6(a+b)}x^3, \qquad \text{(iii)}$$

and the integral of (ii) is

$$EIy = \frac{ab(2a+b)W}{6(a+b)}(a+b-x) - \frac{aW}{6(a+b)}(a+b-x)^3. \qquad \text{(iv)}$$

[The constants are found from the conditions that (i) and (ii) give the same values of y and of dy/dx when $x = a$, and that for (i) $y = 0$ when $x = 0$ and for (ii) $y = 0$ when $x = a + b$.]

7. The deflection y_1 and the gradient y'_1 at D in example 6 are

$$y_1 = \frac{a^2b^2W}{3(a+b)EI}, \qquad y'_1 = \frac{ab(b-a)W}{3(a+b)EI}.$$

8. Show that a in § 68 is the length of a piece of the chain whose weight is equal to the tension at the lowest point or **vertex** of the catenary.

9. Show that the tension at any point of the common catenary is equal to the weight of a piece of the chain whose length is equal to the ordinate at the point.

10. If from M, the foot of the ordinate at P in Fig. 36, a perpendicular is drawn to the tangent at P to meet the tangent at G, show that $MG = a$ and $GP = \text{arc } CP$.

11. The ends A, B of a chain are in the same horizontal and the chain bears a continuously distributed load, which is uniform per foot run of the span (w lb. per foot run). Taking the origin at the lowest

point of the chain and the y-axis vertical (upwards), show that the equation of the curve of the chain is $y=wx^2/2H$, where H is the tension at the lowest point of the chain.

If the span $AB=2b$ and if the depth of the lowest point of the chain below AB is c, show that the tension at B is

$$\frac{wb}{2c}\sqrt{(b^2+4c^2)}.$$

12. If the half length s of the chain in example 11 undergoes a small change δs, say through strain or through change of temperature, show that the amount by which the chain will sag is approximately $3b\delta s/4c$.

[Use the approximate value for s, namely $s=b+2c^2/3b$, given in Exercises IX., example 16.]

13. A cord, whose ends are free, presses tightly against a rough cylinder along an arc AB, the cord lying in a plane which is normal to the axis of the cylinder. If T is the tension at any point P of the cord between A and B and if θ is the angle between the radii OA and OP (O being the point in which the plane of the cord cuts the axis of the cylinder), show that, when slipping is about to take place, T satisfies the equation

$$\frac{dT}{d\theta}+\mu T=0$$

where μ is the coefficient of friction.

If the tension at A is T_1, show that $T=T_1e^{-\mu\theta}$.

69. Example from Electricity. In this example the notation of differentials will be used; thus, if in time δt the current i increases by δi, the increment of i is δi while the differential of i is di, which is equal to (§ 22)

$$\frac{di}{dt}dt.$$

The equations obtained are all, ultimately, *equations between limits*, and the small errors due to writing the differential for the increment disappear. In practical work the differential is frequently used in place of the increment; thus, in §§ 37–40, dx, dy, dm, dW ... are frequently used instead of δx, δy, δm, δW ....

We consider the case of a variable current flowing in a circuit containing resistance and self-induction. At time t seconds let the impressed E.M.F. be E volts, the current i amperes, the resistance R ohms and the inductance L henries, R and L being constant.

The energy communicated to the circuit in a short time

dt is $Eidt$ joules. Of this energy the amount i^2Rdt takes the form of heat while the amount $iLdi$ goes to increase the energy of the magnetic field or, as is sometimes said, to overcome the counter E.M.F. due to self-induction. The energy communicated to the circuit is equal to the sum of these two amounts. Hence we have

$$iLdi + i^2Rdt = Eidt, \quad\quad (1)$$

and therefore, dividing by idt, we obtain

$$L\frac{di}{dt} + Ri = E \quad\quad (2)$$

as the equation connecting the various quantities.

We discuss one or two simple cases, it being understood that E is either a constant or a function of t.

To obtain the integral of (2) divide by L, the coefficient of di/dt; the equation takes the form

$$\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}. \quad\quad (3)$$

Now multiply by $e^{Rt/L}$, where it should be noted that the coefficient of t in the index of e is the coefficient of i in equation (3); we get, as can at once be tested by differentiation,

$$\frac{d}{dt}\left(ie^{\frac{Rt}{L}}\right) = \frac{E}{L}e^{\frac{Rt}{L}}, \quad\quad (4)$$

and therefore, integrating and putting A for the constant of integration,

$$ie^{\frac{Rt}{L}} = A + \int\frac{E}{L}e^{\frac{Rt}{L}}dt = A + f(t) \quad\quad (5)$$

where $f(t)$ is the value of the integral.

Now multiply by $e^{-Rt/L}$; therefore

$$i = Ae^{-\frac{Rt}{L}} + e^{-\frac{Rt}{L}}f(t). \quad\quad (6)$$

Equation (6) is the integral of equation (2) or (3).

Example 1. $E = \text{constant} = E_0$; $i = 0$ when $t = 0$.

In this case $$f(t) = \frac{E_0}{L}\int e^{\frac{Rt}{L}}dt = \frac{E_0}{R}e^{\frac{Rt}{L}},$$

and from (6) $$i = Ae^{-\frac{Rt}{L}} + \frac{E_0}{R}.$$

When $t=0$, $i=0$, and therefore

$$0=A+E_0/R; \quad A=-E_0/R,$$

so that

$$i=\frac{E_0}{R}\left(1-e^{-\frac{Rt}{L}}\right). \qquad (7)$$

If we put $L/R=T$, then T (which, since the index of e must be a pure number, is a quantity of the same kind as t, that is a *time*) is called the **time-constant of the circuit.** If $L=0\cdot01$, $R=0\cdot1$, then $T=\frac{1}{10}$. At the end of $\frac{1}{10}$th of a second the exponential term is e^{-1}, and the current is

$$i=\frac{E_0}{R}(1-e^{-1})=\frac{E_0}{R}(1-0\cdot368)=0\cdot632\,\frac{E_0}{R}.$$

As t increases, the exponential term in (7) soon becomes very small; thus if $t=\frac{1}{4}$ and $T=\frac{1}{10}$, the term $=e^{-2\cdot5}=0\cdot082$. The value of i therefore very rapidly approaches the *steady value* E_0/R.

Example 2. $E=E_0\sin pt$ where E_0 is a constant.

In this case

$$f(t)=\frac{E_0}{L}\int e^{\frac{Rt}{L}}\sin pt\,dt$$

$$=E_0\times\frac{e^{\frac{Rt}{L}}(R\sin pt-pL\cos pt)}{R^2+p^2L^2} \qquad (8)$$

by § 62, example 1.

A more convenient form is obtained by putting

$$R=k\cos\theta, \qquad pL=k\sin\theta,$$

so that

$$k=\surd(R^2+p^2L^2), \quad \tan\theta=pL/R.$$

Equation (8) now becomes

$$f(t)=\frac{E_0}{\surd(R^2+p^2L^2)}e^{\frac{Rt}{L}}\sin(pt-\theta), \qquad (9)$$

and equation (6) becomes

$$i=Ae^{-\frac{Rt}{L}}+\frac{E_0}{\surd(R^2+p^2L^2)}\sin(pt-\theta). \qquad (10)$$

The exponential term decreases very rapidly and the value of i becomes, after a very short time,

$$i=\frac{E_0}{\surd(R^2+p^2L^2)}\sin(pt-\theta). \qquad (11)$$

The *phase*-angle of i, that is the angle $pt-\theta$, is less than that of E by θ, which is called the angle of *lag*. When expressed in terms of time the phase of the current is said to be behind that of the electromotive force.

EXERCISES. XXII.

1. If in § 69, equation (7), i_0 is put for E_0/R and if i_1, i_2, ... denote the values of i for t equal to T, $2T$, ..., show that

$$\frac{i_1}{i_0}=0{\cdot}632,\quad \frac{i_2}{i_0}=0{\cdot}865,\quad \frac{i_3}{i_0}=0{\cdot}950,\quad \frac{i_4}{i_0}=0{\cdot}982.$$

2. Integrate the equation

$$L\frac{di}{dt}+Ri=0,$$

given that $i=i_0$ when $t=0$. Calculate the values i_1/i_0, i_2/i_0, ... where i_1, i_2, ... have the same meaning as in example 1.

3. The current i satisfies the conditions of example **1**, § 69, from $t=0$ to $t=t_1$; at time t_1 the electromotive force E_0 ceases to act, so that i satisfies the equation

$$L\frac{di}{dt}+Ri=0,$$

from $t=t_1$ onwards. Draw the graph of i.

4. If at time t_1, in the case of example 1, § 69, the resistance is suddenly increased to R_1, where R_1 is constant, show that for values of t greater than t_1,

$$i=\frac{E_0}{R_1}\left\{1-e^{-\frac{R_1(t-t_1)}{L}}\right\}+\frac{E_0}{R}\left\{1-e^{-\frac{Rt_1}{L}}\right\}e^{-\frac{R_1(t-t_1)}{L}}.$$

5. Show that for the value of i in example 4 the product iR_1 tends, as t is taken nearer and nearer to t_1, to the value

$$\frac{E_0R_1}{R}\left(1-e^{-\frac{Rt_1}{L}}\right),\quad \text{or simply}\quad \frac{E_0R_1}{R}$$

if Rt_1/L is at all large. (Note, for example, that $e^{-10}=0{\cdot}0000454$.)

6. In the case of example 1, § 69, show that the quantity of electricity, Q, that traverses the circuit in consequence of self-induction from $t=t_1$ to $t=t_2$ is given by the equation

$$Q=-\frac{L}{R}\int_{t_1}^{t_2}\frac{di}{dt}dt=-\frac{L(i_2-i_1)}{R}=-\frac{LE_0}{R^2}\left(e^{-\frac{Rt_1}{L}}-e^{-\frac{Rt_2}{L}}\right).$$

If $t_1=0$ and t_2 is very large show that

$$Q=-\frac{LE_0}{R^2}=-Ti_0,$$

where i_0 is the steady value of the current and T the time-constant.

Give the graphical interpretation of Q.

7. Show that the work done against the electromotive force of self-induction in the case considered in § 69, as the current increases from zero to the value i, is $\frac{1}{2}Li^2$.

8. If $i = I \sin(pt + \alpha)$ show that

$$\frac{di}{dt} = pI \sin\left(pt + \alpha + \frac{\pi}{2}\right).$$

Note that $\cos x = \sin(x + \pi/2)$.

9. Find (i) the average value, (ii) the R.M.S. value of the current i (example 8) for a complete period. (§ 46, example 2.)

10. For the value of i in § 69 (11) show that, if R is very large compared with pL, the value of i is approximately

$$i = \frac{E_0}{R} \sin pt,$$

while if pL is large compared with R the value is approximately

$$i = -\frac{E_0}{pL} \cos pt.$$

11. Show that the energy communicated to the circuit during one period $T (= 2\pi/p)$ in the case of § 69, example 2, is

$$W = \int_0^T Ei\,dt = \frac{T}{2}\,\frac{E_0^2}{\sqrt{(R^2 + p^2L^2)}} \cos\theta,$$

and that the mean value of Ei for the period T is

$$\tfrac{1}{2}\,\frac{E_0^2}{\sqrt{(R^2 + p^2L^2)}} \cos\theta = \frac{W}{T}.$$

12. If $\bar{E}$ and $\bar{i}$ are the R.M.S. values of E and i for the period T show that the mean value W/T in example 11 is

$$\frac{W}{T} = \bar{E} \times \bar{i} \times \cos\theta.$$

13. In § 69, example 2, determine the constant A if $i = 0$ when $t = t_1$ Work out the solution and represent it graphically if $R = 50$, $L = 1$, $p = 1000$, $pt_1 - \theta = \pi/6$. (Bedell and Crehore, *Alternating Currents*, p. 56. London: Whittaker.)

14. A condenser of capacity C is charged till the potential difference (P.D.) of its plates is V_0; the two plates are then connected by a wire of resistance R but of negligible inductance. If at time t the P.D. is V, the charge Q, and the current i, then

$$Q = CV, \quad i = -\frac{dQ}{dt}, \quad V = Ri,$$

and therefore

$$CR\frac{dV}{dt} + V = 0.$$

Show that (i) $V = V_0 e^{-\frac{t}{CR}}$; (ii) $R = \dfrac{t}{C \log(V_0/V)}$.

15. If in example 14 the inductance is L, then

$$Q=CV,\ i=-\frac{dQ}{dt},\ V=Ri+L\frac{di}{dt},$$

and therefore

$$\frac{d^2V}{dt^2}+\frac{R}{L}\frac{dV}{dt}+\frac{1}{CL}V=0.$$

[This equation is of the form given in example 4, § 62. In the notation of that example

$$k=R/L,\ \mu=1/CL.$$

Hence if $4L>CR^2$, V is periodic with decreasing amplitude, while if $4L<CR^2$, V is not periodic.]

70. Double Integrals. The notation of double integrals occurs so frequently in elementary applications of the calculus to mechanics that it seems proper to give an explanation of it. The following problem, though it may be solved more simply otherwise, shows clearly the nature of double integration.

Example. Find the second moment of an ellipse about an axis through its centre perpendicular to its plane.

Let the equation of the ellipse (Fig. 37) be

$$x^2/a^2+y^2/b^2=1. \qquad \text{.................................} (1)$$

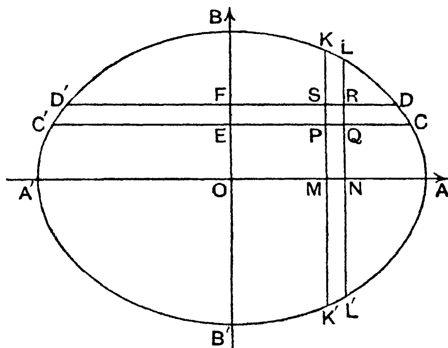


Fig. 37.

Divide the ellipse into a large number n of small rectangles by drawing two sets of straight lines parallel to the axes; portions of the rectangles at the boundary of the ellipse will, of course, lie outside the ellipse.

Let $PQRS$ be a typical rectangle; $OM=x$, $MP=y$, $MN=PQ=\delta x$, $PS=\delta y$, rect. $PQRS=\delta x\delta y=\delta A$.

The second moment of δA is, approximately, $OP^2 . \delta A$ which is equal to $(x^2+y^2)\delta A$, and therefore the second moment of the ellipse is, approximately,

$$\sum(x^2+y^2)\delta A \text{ or } \sum(x^2+y^2)\delta x\delta y \quad \text{.....................(2)}$$

where the summation Σ is to be extended over all the n small rectangles.

The required moment is the *limit* to which the sum (2) converges when n becomes infinite, each rectangle at the same time converging to zero.

To obtain this limit we proceed as follows.

First, let x and δx be kept constant, and sum with respect to y. This operation will give all the terms in (2) that arise from the rectangles in a strip parallel to the y-axis; if $x=OM$, $\delta x=MN$ it will give all the rectangles arising from the strip $K'L'LK$. When δy converges to zero, the limit of the part of (2) thus arising is

$$\delta x\int_{MK'}^{MK}(x^2+y^2)dy. \quad \text{...............................(3)}$$

In finding this integral it must be remembered that x is treated as a constant. The integral will contain x (or OM), MK and MK'. But by equation (1)

$$MK'=-\frac{b}{a}\surd(a^2-x^2),\quad MK=+\frac{b}{a}\surd(a^2-x^2), \quad \text{..............(4)}$$

so that, on the whole, (3) contains x and the constants a, b. For brevity, let

$$\int_{MK'}^{MK}(x^2+y^2)dy=F(x). \quad \text{..........................(5)}$$

The terms of (2) arising from a strip parallel to the y-axis have therefore for limit $F(x)\delta x$.

Secondly. Find now the limit of $\Sigma F(x)\delta x$, that is, find the limit of the sum of all the terms arising from the strips like $K'L'LK$. The limit of this sum is

$$\int_{-a}^{a}F(x)dx \quad \text{.................................(6)}$$

When $F(x)$ is replaced by its value (5), the integral (6) is written thus:

$$\int_{-a}^{a}dx\int_{MK'}^{MK}(x^2+y^2)dy, \quad \text{...........................(7)}$$

which is called **a double integral.**

The mode of deriving (6) shows that (7), which is merely the fuller symbol for (6), means :—*Integrate (x^2+y^2) with respect to y from $y=MK'$ to $y=MK$, treating x as a constant during this integration; then integrate the result with respect to x from $x=-a$ to $x=a$.*

We might also find the limit of (2) by first keeping y and δy constant. Instead of (3) we should have

$$\delta y\int_{EC'}^{EC}(x^2+y^2)dx, \quad \text{...........................(8)}$$

which arises from the strip $C'CDD'$, and instead of (7) we should have

$$\int_{-b}^{b} dy \int_{EC'}^{EC} (x^2+y^2)dx. \qquad \text{.........................(9)}$$

In (9) we first integrate as to x, *keeping y constant*, from $x=EC'$ to $x=EC$ where by (1)

$$EC' = -\frac{a}{b}\sqrt{(b^2-y^2)}, \quad EC = +\frac{a}{b}\sqrt{(b^2-y^2)},$$

and then we integrate the result as to y from $y=-b$ to $y=b$.

The evaluation of (7) or (9) is not hard; take (7), then

$$\int_{MK'}^{MK} (x^2+y^2)dy = \Big[x^2y+\tfrac{1}{3}y^3\Big]_{MK'}^{MK} = \frac{2b}{a}\left\{x^2+\frac{b^2}{3a^2}(a^2-x^2)\right\}(a^2-x^2)^{\frac{1}{2}}. \text{ (10)}$$

We have now to integrate with respect to x from $x=-a$ to $x=a$. Since the integrand contains only *even* powers of x, the integral will be *twice* the integral from $x=0$ to $x=a$. To integrate, put $x=a\sin u$ and we find

$$\int_{-a}^{a} \frac{2b}{a}\left\{x^2+\frac{b^2}{3a^2}(a^2-x^2)\right\}(a^2-x^2)^{\frac{1}{2}}\,dx$$

$$=4ab\int_0^{\frac{\pi}{2}} (a^2\sin^2 u\cos^2 u+\tfrac{1}{3}b^2\cos^4 u)\,du$$

$$=\pi ab\,.\,\frac{a^2+b^2}{4}. \qquad \text{...(11)}$$

The area of the ellipse is πab, and therefore the square of the radius of gyration is $\frac{1}{4}(a^2+b^2)$.

Instead of the symbols (7), (9) the form

$$\iint (x^2+y^2)\,dx\,dy \qquad \text{..............................(12)}$$

is often used. To make this symbol quite definite some phrase must be used to indicate the limits; for example, "the integration being extended over the area of the ellipse," the particular method of effecting the integrations being left unstated.

Enough has been said to indicate the general meaning of a double integral; for further details see the author's *Calculus*, §§ 135, 136.

CHAPTER XIV.

GRAPHICAL INTEGRATION.

71. Graphical Construction of Derived Curve. In § 20 the meaning of the expression "derived curve" has been explained. We now state a graphical method of tracing the derived curve of a given curve.

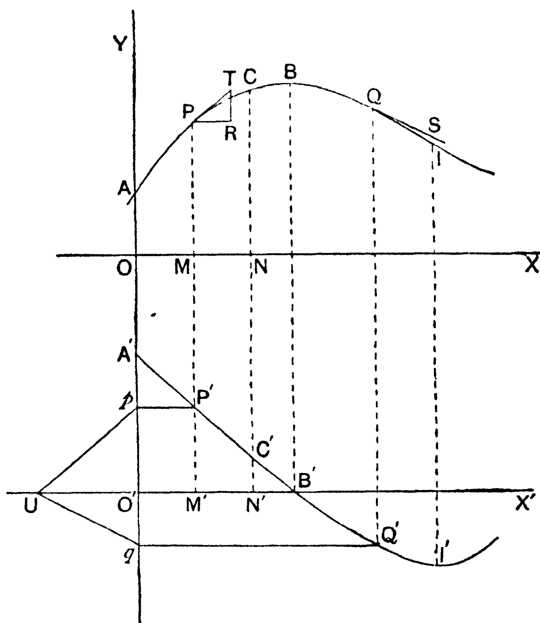


Fig. 38.

Let APB (Fig. 38) be the given curve, referred to OX, OY as axes of coordinates. Through any convenient point

O' on the y-axis draw $O'X'$ parallel to OX, and take $O'X'$, $O'Y$ as the axes for the derived curve.

Let PT be the tangent at any point P on the given curve, and let $OM=x$, $MP=y=f(x)$ be the coordinates of P. If PR is parallel to OX and RT perpendicular to PR, then

$$\frac{dy}{dx}=\tan RPT=\frac{RT}{PR}.$$

Now take a point U on $O'X'$ to the left of O' and draw Up parallel to PT to meet the common axis of ordinates at p. The triangles $UO'p$ and PRT are similar; therefore

$$\frac{O'p}{UO'}=\frac{RT}{PR}=\frac{dy}{dx},\quad O'p=\frac{dy}{dx}\cdot UO'.$$

Produce PM to cut $O'X'$ at M' and draw pP' parallel to $O'X'$ to meet PM' at P'; then

$$M'P'=O'p=\frac{dy}{dx}\cdot UO'.$$

If the unit segment for the ordinates of the derived curve is taken equal to UO', then

$$M'P'=\frac{dy}{dx},$$

so that $M'P'$ is the ordinate of the derived curve corresponding to the point P on the given curve. The abscissa $O'M'$ of P' is equal to the abscissa OM of P, and P, P' may be called corresponding points, MP, $M'P'$ corresponding ordinates.

To find the point corresponding to any other point Q on the given curve, draw Uq parallel to the tangent QS to meet the y-axis at q and then draw qQ' parallel to $O'X'$ to meet the ordinate of Q (produced) at Q'; Q' is the point corresponding to Q.

In this way we can find any number of points on the derived curve and then, by drawing a fair curve through these points, the derived curve itself.

72. Integral Curve. If we denote any ordinate of the derived curve by y', so that $y'=f'(x)=dy/dx$ where y or $f(x)$ is the corresponding ordinate of the given curve, then the

area $O'M'P'A'$ bounded by the derived curve, the two axes and the ordinate $M'P'$ is equal to

$$\int_0^{O'M} y'dx.$$

But this integral is equal to $f(x)-f(0)$, that is, to $MP-OA$. In other words, the area bounded by the derived curve, the two axes of coordinates of the derived curve, and the ordinate at any point of the derived curve is equal to the corresponding ordinate of the given curve diminished by the ordinate at the origin of the given curve.

In the same way we see that the area $M'N'C'P'$ is equal to the ordinate at C diminished by the ordinate at P, namely $NC-MP$.

With respect to the derived curve, therefore, the given curve is such that the difference between any two of its ordinates is equal to the area bounded by the corresponding ordinates of the derived curve, the x-axis of the derived curve, and the derived curve itself. From this property the given curve is called the **integral curve** of the derived curve.

Of the two curves APC, $A'P'C'$ either may now be considered to be the given curve; if APC is considered as given, then $A'P'C'$ is its derived curve, while if $A'P'C'$ is considered as given then APC is its integral curve.

In the expression for the area, namely $MP-OA$, the term $-OA$ is the constant of integration; if the axis OX be drawn through A this term will disappear. In practice the origin is usually chosen so that the integral curve passes through it; the ordinate MP then measures the area.

Before stating a graphical construction for the integral curve we shall refer to the question of units (see also § 32). Expressing $M'P'$ explicitly in terms of UO', we have

$$M'P'=\frac{dy}{dx}\,.\,UO', \text{ so that } \int M'P'dx=y\,.\,UO'.$$

For both curves the x-scale is the same, say a inches $=1$. Let the y-scale of the integral curve APC be b inches $=1$. The scale for the ordinates of the curve $A'P'C'$ is $UO'=1$; let $UO'=n$ inches.

The value $y\,.\,UO'$ therefore represents $y\times bn$ square

inches; $y=1$ gives bn sq. in., $y=7$ gives $7bn$ sq. in. and so on. If, for instance, the y-scale is $1''=20$ and if $UO'=\frac{3}{4}''$, then $b=1/20$, $n=3/4$ and the value $y=7$ represents $7\times\frac{1}{20}\times\frac{3}{4}$, that is $\frac{21}{80}$ths of a square inch.

If all the values are expressed in inches, then the number of inches in the ordinate of the integral curve multiplied by the number of inches in UO' will give the number of square inches in the area, when the integral curve goes through the origin; if the integral curve does not go through the origin, it will be necessary to subtract the corresponding value of OA, the ordinate at the origin.

We have chosen the unit for the y-scale of the curve $A'P'C'$ to be UO'; it is easy however to make the necessary changes when the unit is differently chosen.

73. Graphical Construction of Integral Curve. The following method of constructing an integral curve is based on that of § 71 for drawing the derived curve.

On the x-axis $O'X'$ of the curve whose integral curve is to be drawn, lay off from O' equal short segments $O'1_1$, $1_1 2_1$, $2_1 3_1$, ... and through the points 1_1, 2_1, 3_1, ... draw parallels to the common y-axis. (Fig. 39.)

Let the ordinates at 2_1, 4_1, 6_1, ... cut the given curve at $2'$, $4'$, $6'$, ... and let $2''$, $4''$, $6''$, ... be the projections of these points on the y-axis; $0'$ is the point at which the curve cuts the y-axis.

Let us take the integral curve that passes through O; then the tangent to it at O is parallel to $U0'$. Draw this tangent and let it meet the ordinate through 1_1 at 1.

The tangent at the point on the integral curve corresponding to $2'$ is parallel to $U2''$. Draw 13 parallel to $U2''$ cutting $2_1 2'$ at 2 and meeting the ordinate drawn through 3_1 at 3; 2 is the point corresponding to $2'$.

In the same way draw 35 parallel to $U4''$, cutting $4_1 4'$ at 4 and the ordinate through 5_1 at 5; 4 is the point corresponding to $4'$.

The construction may be repeated and we thus get a series of lines $O1$, 13, 35, ... which are, approximately, tangents to the integral curve, the points O, 2, 4, ... being their points of contact.

The point O from which the construction begins is, of course, arbitrary but when that is fixed the integral curve is determinate. The position of the other points 2, 4, ... is approximate; a discussion of the nature of the approximation is given in the author's *Calculus*, § 83.

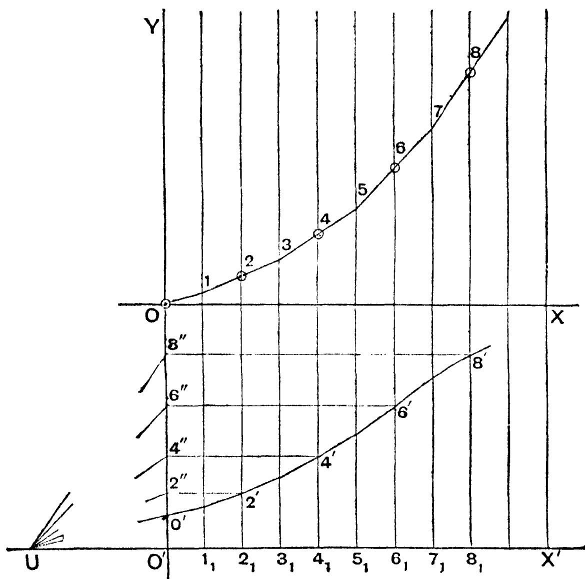


Fig. 39.

The lines through 1_1, 3_1, 5_1, ... parallel to the y-axis are often called **median lines** and the lines UO', $U2''$, $U4''$, ... **direction lines.**

It is difficult to draw a derived curve with any considerable degree of accuracy, because it is difficult to draw accurately the tangent at a point on a curve. It is, however, easy to obtain great accuracy in drawing an integral curve.

The integral curve of the integral curve is called the **second integral curve** of the given one; the integral curve of the second integral curve is called the **third integral curve** of the given one, and so on.

If P is any point on a given curve and P', P'', ... the

corresponding points on the first, second, ... integral curves, and if y, y_1, y_2, ... are the ordinates at P, P', P'', ... respectively, then the gradient of the first integral curve at P' is y, the gradient of the second integral curve at P'' is y_1, and so on. This property of corresponding ordinates is constantly used in applications.

The integral curve of a straight line $y=b$, parallel to the x-axis, is the straight line $y_1=bx$, whose gradient is b.

The integral curve of a straight line $y=ax+b$, not parallel to the x-axis, is the parabola $y_1=\frac{1}{2}ax^2+bx$.

In each case the integral curve has been supposed to pass through the origin; of course, if it does not pass through the origin there will be a constant term.

74. First Moments. We shall now apply the method of graphical integration to the determination of first and second moments of plane areas; the area is supposed to be bounded by a curve, the coordinate axes and an ordinate. We begin with first moments, and take the problem of finding the first moment of the area $OABP$ (Fig. 40) about any axis in its plane perpendicular to OA.

Draw $O'P'B'$ the integral curve of OPB and let $M'P'$, $N'Q'$ be the ordinates corresponding to MP, NQ respectively. By the fundamental property of the integral curve, the areas OMP, ONQ are equal respectively to the ordinates $M'P'$, $N'Q'$, and therefore the area of the strip $MNQP$ is equal to $N'Q'-M'P'$.

If y denote any ordinate of the given curve we shall indicate by y_1 the corresponding ordinate of the first integral curve, by y_2 the corresponding ordinate of the second integral curve and so on; thus if $MP=y$ then $M'P'=y_1$. We shall also use the notation of differentials dx, dy, dy_1, ... instead of that of increments δx, δy, δy_1, ... (See § 69 and § 22.)

Now let

$$OM=x=O'M',\quad MN=dx=M'N',\quad MP=y,\quad M'P'=y_1;$$

then $N'Q'-M'P'=dy_1$.

The area of the strip $MNQP$ is equal to ydx; but this area is also, as we have seen, equal to $N'Q'-M'P'$. Therefore we have the important result

$$ydx=dy_1.$$

Similarly, we have $y_1dx = dy_2$, $y_2dx = dy_3$, and so on; these results are in fact mere consequences of the definitions of the curves.

(i) Let the axis of moments be AB.

Through P', Q' draw parallels to $O'A'$ meeting AB at E', F' and OY at H', K' respectively. Let $OA = a$; then $MA = a - x = P'E'$.

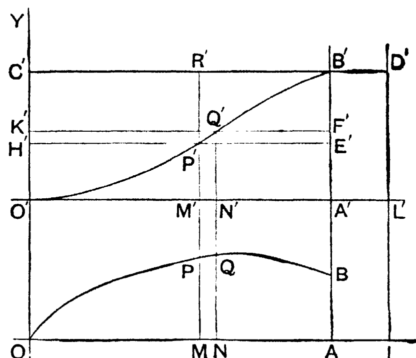


Fig. 40.

The moment of the strip $MNQP$ about AB may be taken as $MA \, . \, ydx$, or $(a-x)ydx$, which is equal to $(a-x)dy_1$. But $(a-x)dy_1$ is equal to $P'E' \, . \, E'F'$ which may be taken as the area of the strip $P'E'F'Q'$. Therefore the moment of the strip $MNQP$ about AB is equal to the area of the strip $P'E'F'Q'$. Hence the moment of the whole area $OABP$ about AB is the sum of the strips like $P'E'F'Q'$, that is, is equal to the area $O'A'B'P'$ of the first integral curve.

(Note that, when the area $OABP$ is divided into strips by a series of ordinates, to each of these strips like $MNQP$ will correspond a strip of the area $O'A'B'P'$ like $P'E'F'Q'$; integration for the first moment is reduced to integration of the strips of the area $O'A'B'P'$. Similar correspondences will be met with repeatedly in this method of graphical integration.)

(ii) Let the axis of moments be OY.

The moment of the strip $MNQP$ about OY is $x \, . \, ydx$, or

xdy_1, which is equal to the area of the strip $H'P'Q'K'$ of the area $O'P'B'C'$, where C' is the point on OY at which the parallel through B' to $A'O'$ meets OY. Hence the moment of the whole area $OABP$ about OY is equal to the area $O'P'B'C'$.

(iii) Let the axis of moments be $LL'D'$, any line perpendicular to OA and not intersecting the area $OABP$.

If $O'A'$, $C'B'$ are produced to meet $LL'D'$ at L', D' it is easily seen that the required moment is the area $O'L'D'B'P'$; the proof is precisely the same as for case (i).

(iv) Let the axis of moments be MP, any line perpendicular to OA and intersecting the area $OABP$.

The moment of the area OMP about MP is by (i) equal to the area $O'M'P'$; the moment of the area $MABP$ about MP is by (ii) equal to the area $PB'R'$, where R' is the point at which MP meets $C'B'$. Therefore the required moment is the *difference* of these areas.

75. Various Constructions. The moment in case (i) of §74 is the area $O'A'B'P'$ of the first integral curve, and is therefore equal to the ordinate of the second integral curve corresponding to $A'B'$ or AB. For the sake of clearness we shall draw the second integral curve in a separate diagram, retaining $O'A'$ as the x-axis; points on the second integral curve will be denoted by letters with two accents: thus, P'' corresponds to P' on the first integral curve and to P on the given curve. (Fig. 41.)

(a) **Centroid of Area.** Let the tangent at B'' to the second integral curve cut $O'A'$ at G'' and $O'Y$ at V''; the centroid of the area $OABP$ will lie on the ordinate through G''.

From the figure, the gradient at B'' is $A'B''/G''A'$; but the gradient at B'' is also equal to $A'B'$, the ordinate at B' on the first integral curve. Hence

$$A'B' = \frac{A'B''}{G''A'}, \text{ or } G''A' = \frac{A'B''}{A'B'}.$$

But $A'B'$ is equal to the area $OABP$, and $A'B''$ is equal to the moment of that area about $A'B'$, so that the centroid lies on the ordinate through G''.

(*b*) **Moment of Area about** OY. In case (ii), § 74, the moment is equal to the area $O'P'B'C'$ (Fig. 40), that is, to the rectangle $O'A'B'C'$ diminished by the area $O'A'B'P'$.

Now, the integral curve of the straight line $C'B'$ is the straight line through O' parallel to the tangent $G''B''$; if this straight line meet $A'B''$ produced at b'' (not shown in figure) then $A'b''$ will be equal to the rectangle $O'A'B'C'$. From $A'b''$ subtract $A'B''$, which is equal to the area $O'A'B'P'$, and we obtain $B''b''$ as the line which represents the area $O'P'B'C'$; but $B''b''$ is equal to $V''O'$, and therefore $V''O'$ represents the area $O'P'B'C'$, that is, the moment of $OABP$ about OY.

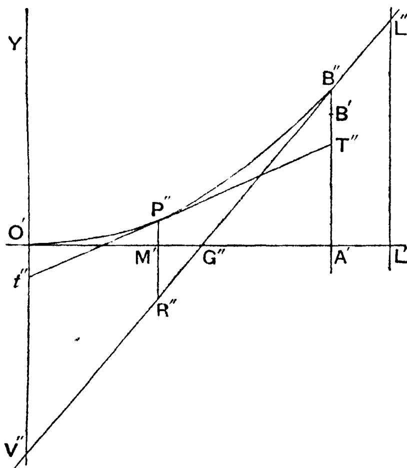


Fig. 41.

(*c*) **Moment of Area about** LL'. Let $G''B''$ be produced to meet the axis LL' at L'', and let the integral curve $O'b''$ of $C'B'$ be produced to meet LL' at l'' (not shown in figure). The line $L'l''$ will be equal to the area of the rectangle $O'L'D'C'$. Also, $L''l''$ is equal to $V''O'$ which, as we have just seen, is equal to the area $O'P'B'C'$; therefore $L'l''-L''l''$, that is, $L'L''$ is equal to the area $O'L'D'B'P'$. Hence $L'L''$ is equal to the moment of $OABP$ about LL'.

(*d*) **Moment of Area about** MP. This moment is the

difference of the areas $O'M'P'$ and $P'B'R'$ (Fig 40). Suppose $O'M'P'$ less than $P'B'R'$; the moment is then $P'B'R' - O'M'P'$.

Now $O'M'P' = M'P''$ (Fig. 41). Also $P'B'R'$ is equal to the rectangle $M'A'B'R'$ diminished by $M'A'B'P'$. But the rectangle is equal to $M'A' \times A'B'$, and $M'A'B'P'$ is equal to $A'B'' - M'P''$. Therefore

$$P'B'R' - O'M'P' = M'A' \times A'B' - (A'B'' - M'P'') - M'P''$$

$$= M'A' \times \frac{A'B''}{G''A'} - A'B'', \text{ since } A'B' = \frac{A'B''}{G''A'}$$

$$= \frac{A'B''}{G''A'}(M'A' - G''A') \text{ or } \frac{A'B''}{G''A'} \cdot M'G''.$$

If $P''M'$ be produced to meet $V''B''$ at R'', then

$$A'B'' : G''A' = R''M' : M'G'',$$

so that

$$P'B'R' - O'M'P' = R''M'.$$

When the axis passes through G'' the moment is zero. We have thus another proof that the centroid of the area lies on the ordinate through G''.

General Rule. By examining the various cases we see that the moment of the area $OABP$ about any axis perpendicular to OA is equal to the line intercepted on the axis between $O'A'$ and the tangent $V''B''$ to the second integral curve at B''.

(e) **Moment of Strip** $MNQP$ **about** AB **and about** OY. Let the tangent at P'' meet $A'B''$ at T'' and $O'Y$ at t'' (Fig. 41). Then, by the general rule applied to the area OMP, the moment of OMP about AB is $A'T''$. If the tangent at Q'' (Fig. 42) meet $A'B''$ at T''_1, then the moment of ONQ about AB is $A'T''_1$. Hence the moment of the strip $MNQP$ (the difference of ONQ and OMP) about AB is $T''T''_1$ (Fig. 42).

Similarly it will be seen that if the tangent at Q'' meet $O'Y$ at t''_1 the moment of the strip $MNQP$ about OY is t''_1t''.

76. Second Moments. We shall find the second moment of the area $OABP$ (Fig. 40) about any axis perpendicular to OA.

The second moment about AB is, the previous notation being retained,

$$\int_0^a (a-x)^2 y\,dx.$$

Integrating by parts, we find

$$\int_0^a (a-x)^2 y dx = \Big[(a-x)^2 y_1\Big]_0^a + 2\int_0^a (a-x) y_1 dx.$$

But $(a-x)^2=0$ when $x=a$, and $y_1=0$ when $x=0$, so that the integrated part is zero. Integrating again, we find

$$\int_0^a (a-x)^2 y dx = \Big[2(a-x) y_2\Big]_0^a + 2\int_0^a y_2 dx = 2\int_0^a y_2 dx.$$

The moment is therefore twice the area $O'A'B''$ of the second integral curve, and is therefore equal to $2A'B'''$ where $A'B'''$ is the ordinate of the third integral curve corresponding to AB.

We have given this proof because of its simplicity; we shall however discuss the various cases in a manner similar to that adopted in §§ 74, 75.

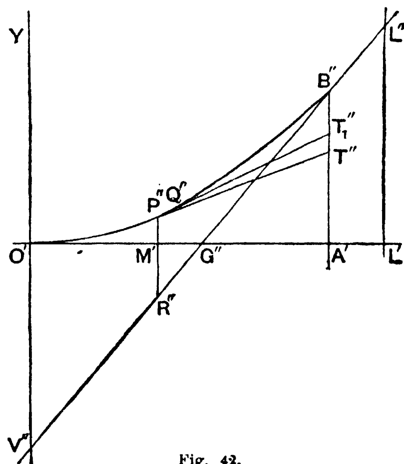


Fig. 42.

(i) Let the axis of the moments be AB.

The second moment of the strip $MNQP$ about AB is $(a-x)^2 y dx$, that is $(a-x).(a-x)y dx$. But $(a-x)y dx$ is the first moment of the strip about AB, and is therefore, by § 75 (e), $= T''T''_1$.

The area (Fig. 42) bounded by the arc $P''Q''$, the tangents $P''T''$, $Q''T''_1$, and the line $T''T''_1$ may be taken as a triangle

with vertex P'' and base $T'''T''_1$ (any error due to this approximation will not affect the limit of the summation which gives the moment of the whole area $OABP$). The altitude of this triangle is $a-x$, so that $(a-x)\times T'''T''_1$ is equal to twice its area.

Hence the second moment of the strip $MNQP$ about AB is twice the area $P'''T'''T''_1Q''$.

Proceeding now as in §74 (i) (see the remark made at that place) we find that the second moment of the whole area $OABP$ about AB is equal to twice the area $O'A'B''$ of the second integral curve.

(ii) Let the axis pass through the centroid of the area $OABP$.

Let I_G denote the second moment about the axis through G'' parallel to AB; we know [§75 (a)] that this axis passes through the centroid. Then, by §39, Theorem 2, I_G is equal to the moment about AB diminished by the product of the area $OABP$ and $G''A'^2$.

But the area is equal to $A'B'$, and $A'B'\times G''A'$ is equal to $A'B''$; therefore

$$\begin{aligned} I_G &= 2O'A'B''P'' - G''A'\times A'B'' \\ &= 2O'A'B''P'' - 2\Delta G''A'B'' \\ &= 2O'G''B''P''. \end{aligned}$$

(iii) Any axis perpendicular to OA.

Let the axis be, for example, MP or $M'P''$; then the moment is equal to $I_G + A'B'\times M'G''^2$. But $A'B'\times M'G''$ is equal to $R''M'$, and therefore $A'B'\times M'G''^2$ is equal to $R''M'\times M'G''$, that is, to twice the triangle $M'G''R''$. Hence the moment about MP is equal to twice the area $O'G''B''P''$ together with twice the triangle $M'G''R''$.

Similarly the moment about OY is twice the area $O'G''B''P''$ together with twice the triangle $O'G''V''$; the moment about LL' is twice the area $O'L'L''B''P''$.

77. Construction for Second Moment. Let the integral curve of the straight line $V''G''B''$ be constructed with G'' as origin, and let B''' be the point on the integral curve of $O'P''B''$ (that is, on the third integral curve of OPB) corresponding to B''. (Fig. 43.)

The ordinate $A'B'''$ is equal to the area $O'A'B''$, and the

ordinate $A'b'''$ is equal to the triangle $G''A'B''$. Hence $b'''B'''$ is equal to the area $O'G''B''P''$, that is, to $\frac{1}{2}I_G$.

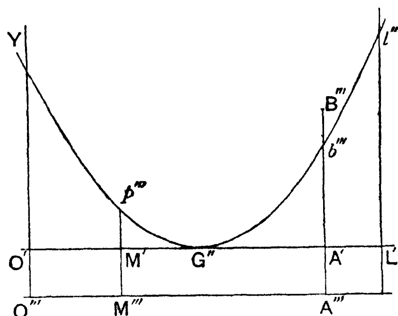


Fig. 43.

Draw $O'''A'''$ parallel to $O'A'$, at a distance below $O'A'$ equal to $b'''B'''$; then the second moment of the area $OABP$ about any axis is twice the line intercepted on the axis between $O'''A'''$ and the parabola $p'''G''b'''$. For example the second moment about $M'''p'''$ is $2M'''p'''$.

78. References. The method of graphical integration has in recent years been extensively applied in the treatment of problems in mechanics, but the discussion of its various uses would lead us far beyond the scope of a mathematical text-book. It is hoped however that enough has been said to enable the student to apply the method in his further studies. A few references are here given.

The standard work is the book of M. Abdank-Abakanowicz, *Les Intégraphes; la courbe intégrale et ses applications* (Paris: Gauthier-Villars), of which there is a German translation by Bitterli, *Die Integraphen* (Leipzig: Teubner). This work contains a full description of the Integraph—an instrument for tracing integral curves.

An article by Prof. W. F. Durand in the *Sibley Journal of Engineering*, January, 1897, and an article by Mr. John G. Johnstone, *The Uses of the Integraph in Ship Calculations*, in the *Transactions of the Institution of Engineers and Shipbuilders in Scotland*, March, 1904, will also be found useful.

CHAPTER XV.

MACLAURIN'S THEOREM. DIFFERENTIAL EQUATIONS.

79. Infinite Series. In § 44 it has been proved that $\sin x$ is approximately equal to a sum of powers of x, and a limit for the error incurred by stopping at any particular term is given. The structure or "law" of the terms is such that any named term can be written down because the n^{th} or "general" term is $(-1)^{n-1}x^{2n-1}/(2n-1)!$; the series of terms can therefore be continued to any extent. When n, the number of terms, tends to infinity the error tends to zero (p. 116), and it is customary to say

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \text{ to infinity.} \qquad \dots\dots\dots(1)$$

The expression on the right of (1) is called an **infinite series**, and it is said that the series "represents" $\sin x$.

Of course it is impossible actually to calculate and add up an infinite number of terms; but when the law of the terms is known and when, as in this example, all the terms that follow a definite term, say the 10^{th} or the 100^{th}, contribute a demonstrably small amount, the infinite series is quite intelligible. Infinite series are of very great use; for instance, see p. 166, where it is shown how logarithms may be calculated. We shall consider very briefly some rules for the use of infinite series.

80. Convergence. Divergence. Let $u_1, u_2, u_3, \dots$ be an endless set or sequence of numbers, and let s_n denote the sum of the first n terms,

$$s_n = u_1 + u_2 + \dots + u_n.$$

When n tends to infinity the expression $u_1+u_2+\dots+u_n$ becomes an infinite series; if, as n tends to infinity, s_n tends to a definite, finite limit, s say, the series is said to **converge** or to be **convergent** and to have **the sum** s or to **converge** to s, but if s_n tends to infinity the series is said to **diverge** or to be **divergent.**

If r_n denote the series formed by beginning at the $(n+1)^{\text{th}}$ term,

$$r_n=u_{n+1}+u_{n+2}+\dots \text{ to infinity},$$

r_n is called "the remainder after n terms." When the series converges, r_n, which is equal to $s-s_n$, must tend to zero when n tends to infinity; r_n will be small when n is sufficiently large. If r_n is small when n is small the series is said to be *rapidly* or *highly* convergent; for purposes of calculation rapid convergence is obviously a valuable quality.

Example. If $u_n=ar^{n-1}$ the series is a geometrical progression (G.P.). In this case

$$s_n=a+ar+ar^2+\dots+ar^{n-1}=\frac{a}{1-r}-\frac{ar^n}{1-r}.$$

(i) If r is numerically less than unity r^n tends to 0 when n tends to infinity; the infinite series is therefore convergent and its sum s is $a/(1-r)$. In this case $r_n=ar^n/(1-r)$.

(ii) If r is numerically greater than unity r^n tends to infinity; therefore s_n tends to infinity and the series is divergent.

It may happen that s_n does not tend to a *definite*, finite limit; thus, if in the G.P. $r=-1$, we have

$$s_n=a-a+a-a+\dots \text{ to } n \text{ terms},$$

and $s_n=2a$ or 0 according as n is even or odd. In this case the series is said to **oscillate**; s_n does not become infinite, but it has no *definite* limit, and is therefore not convergent.

Obviously a series cannot oscillate when all the terms are of the same sign. If the terms are all positive s_n will increase as n increases, and then one of two things must happen. *Either* s_n will increase beyond all bound as n tends to infinity [*e.g.* if $s_n=1+2+3+\dots+n=\frac{1}{2}n(n+1)$], in which case the series will diverge, *or else*, however great n may be taken, s_n will never exceed some fixed number,

k say; in this case we will assume that s_n will tend, as n tends to infinity, to a *definite* limit which will not exceed k, and the series therefore will converge.

We shall represent an infinite series by such notations as

$$u_1+u_2+\ldots; \quad \sum_1^\infty u_n; \quad \sum u_n; \quad \sum u.$$

Note. In testing whether a series is convergent or not, we may neglect any *finite* number of terms; such terms would not affect the character of the series for n tending to infinity.

81. Ratio Test. The following test or rule is very important; it may be referred to as the Ratio Test.

Let $u_1+u_2+\ldots$ be a series of *positive* terms such that when n tends to infinity the ratio u_{n+1}/u_n tends to the definite number r; the series will converge if $r<1$, but will diverge if $r>1$. The test fails to decide whether the series converges or diverges if $r=1$.

(i) $r<1$. Since the limit of u_{n+1}/u_n is r the ratio will differ very little from r *when n is sufficiently large*, and will therefore be less than some number, ρ say, which may be greater than r but will be less than unity. Let u_{n+1}/u_n be less than ρ for every value of n that is equal to or greater than the fixed number m; then

$$u_{m+1}/u_m<\rho \quad \text{or} \quad u_{m+1}<u_m\rho,$$

$$u_{m+2}<u_{m+1}\rho<u_m\rho^2, \quad u_{m+3}<u_{m+2}\rho<u_m\rho^3,$$

and so on. Hence each term, beginning with u_{m+1}, is less than the corresponding term of the G.P. $u_m\rho+u_m\rho^2+\ldots$, which, since $\rho<1$, has the sum $u_m\rho/(1-\rho)$. Therefore the sum s_n of any number n of the terms of the given series, however great n may be taken, is less than the fixed number k, where

$$k=u_1+u_2+\ldots+u_m+u_m\rho/(1-\rho),$$

and therefore, by the assumption stated towards the end of § 80, s_n will tend to a *definite* limit and the series will converge.

(ii) $r>1$. With the same notation as in (i) we now have $\rho>1$, and each term, beginning with u_{m+1}, is greater

than the corresponding term of a G.P. which tends to infinity. The series therefore will diverge.

Note. In the examples the limit L is the limit for n tending to infinity.

Example 1. $$\frac{2}{3}+2\left(\frac{2}{3}\right)^2+3\left(\frac{2}{3}\right)^3+4\left(\frac{2}{3}\right)^4+\ldots.$$

Here $$\frac{u_{n+1}}{u_n}=(n+1)\left(\frac{2}{3}\right)^{n+1}\div n\left(\frac{2}{3}\right)^n=\left(1+\frac{1}{n}\right)\frac{2}{3},$$

and therefore $\mathrm{L}u_{n+1}/u_n=\frac{2}{3}$. Thus $r=\frac{2}{3}<1$ and the series converges. (We might take $m=4$, $\rho=\frac{8}{9}$ or $m=5$, $\rho=\frac{5}{6}$, etc.; usually m is not at all large.)

Example 2. The series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots$ is called the *harmonic series*; show that it diverges.

Here $$\mathrm{L}\frac{u_{n+1}}{u_n}=\mathrm{L}\frac{n}{n+1}=\mathrm{L}\frac{1}{1+1/n}=1,$$

and the Ratio Test fails since $r=1$. But we can prove the result as follows: Beginning with the third term, take the terms in groups of 2, 4 or 2^2, 8 or 2^3, ... 2^m ... ; then

$$\tfrac{1}{3}+\tfrac{1}{4}>\tfrac{1}{4}+\tfrac{1}{4} \text{ or } \tfrac{1}{2},\quad \tfrac{1}{5}+\tfrac{1}{6}+\tfrac{1}{7}+\tfrac{1}{8}>\tfrac{4}{8} \text{ or } \tfrac{1}{2},$$
$$\tfrac{1}{9}+\tfrac{1}{10}+\ldots+\tfrac{1}{16}>\tfrac{8}{16} \text{ or } \tfrac{1}{2},$$

and so on. The sum of p of these groups is $\frac{1}{2}p$, which may be made as big as we please because the number of groups is unlimited. Hence the given series diverges.

This example is noteworthy because $u_n=1/n$, and therefore tends to zero as n tends to infinity. It is easy to see that **a series Σu cannot converge unless** $\mathrm{L}u_n=0$; for if $\mathrm{L}s_n=s$ so also is $\mathrm{L}s_{n-1}$, and, since $u_n=s_n-s_{n-1}$,

$$\mathrm{L}u_n=\mathrm{L}(s_n-s_{n-1})=s-s=0.$$

The example, however, shows that $\mathrm{L}u_n$ may be zero and yet that the series may diverge.

82. Absolute Convergence. If a series which contains both positive and negative terms is convergent when all the negative terms have their signs changed, it will be convergent as it stands. For, obviously, both the sum of the first n terms and the remainder of the given series are numerically less than the corresponding sum and remainder of the series of positive terms, and as the latter remainder tends to zero so does the remainder of the given series, which is therefore convergent.

A series is said to be **absolutely** convergent or **unconditionally** convergent when the series formed from it by making all its terms positive is convergent; any other convergent series is said to be **conditionally** convergent.

The series $1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots$ is absolutely convergent because the series $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots$ has the sum $\pi^2/6$ [p. 149 or § 83, Ex. 8]; but the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$, which has the sum $\log 2$ [p. 166, (v)], is only conditionally convergent since the harmonic series is divergent.

Absolutely convergent series are specially useful; they may be treated in most respects exactly like finite series.

Extension of the Ratio Test. A series will be absolutely convergent if $\mathrm{L}u_{n+1}/u_n$ is *numerically* less than unity.

Note. The *numerical* value of a number, x say, occurs so often that the symbol $|x|$ is used to denote it; $|x|$ means x if x is positive, but $-x$ if x is negative. Thus

$$|5|=5, \quad |-5|=5, \quad |\cos 120°|=|-\tfrac{1}{2}|=\tfrac{1}{2}.$$

83. Examples. The following examples include many of the more important cases. The limit L is the limit for $n=\infty$.

1. *The Binomial Series.* $1+\frac{m}{1}x+\frac{m(m-1)}{1.2}x^2+\ldots.$

$$\frac{u_{n+1}}{u_n}=\frac{m-n+1}{n}x=\left(\frac{m+1}{n}-1\right)x; \quad \mathrm{L}\frac{u_{n+1}}{u_n}=-x.$$

Hence the series converges absolutely if $|x|<1$, diverges if $|x|>1$. (If m is a positive integer the series terminates, and the question of convergence does not arise.)

2. *The Logarithmic Series.* $x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots.$

$$\frac{u_{n+1}}{u_n}=-\frac{n}{n+1}x=-\frac{x}{1+1/n}; \quad \mathrm{L}\frac{u_{n+1}}{u_n}=-x.$$

Hence the series converges absolutely if $|x|<1$, diverges if $|x|>1$. We know (§ 82) that it converges conditionally if $x=1$ and diverges if $x=-1$.

3. *The Exponential Series.* $1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots.$

$$\frac{u_{n+1}}{u_n}=\frac{x}{n}; \quad \mathrm{L}\frac{u_{n+1}}{u_n}=0.$$

Hence the series converges absolutely whatever number x may be.

4. *The Sine Series.* $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$

$$\frac{u_{n+1}}{u_n} = -\frac{x^2}{2n(2n+1)}; \quad \mathrm{L}\frac{u_{n+1}}{u_n} = 0.$$

Hence the series converges absolutely whatever x may be.

5. *The Cosine Series.* $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

$\mathrm{L}u_{n+1}/u_n = 0$, and the series converges absolutely for every x.

6. $x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\frac{x^7}{7} + \dots$

$$\frac{u_{n+1}}{u_n} = \frac{(2n-1)^2 x^2}{2n(2n+1)} = \frac{1 - \frac{1}{n} + \frac{1}{4n^2}}{1 + \frac{1}{2n}} x^2; \quad \mathrm{L}\frac{u_{n+1}}{u_n} = x^2.$$

Hence the series converges absolutely if $|x| < 1$, diverges if $|x| > 1$.

7. Let $u_1 + u_2 + \dots$ (or Σu) and $a_1 + a_2 + \dots$ (or Σa) be two series of *positive* terms. If (i) each term of Σu is less than the corresponding term of Σa, then Σu converges if Σa converges; but if (ii) each term of Σu is greater than the corresponding term of Σa, then Σu diverges if Σa diverges.

(i) Let Σa converge and have k as its sum. The sum of the first n terms of Σa is less than k whatever number n may be; but the sum of the first n terms of Σu is less than the corresponding sum for Σa, and therefore less than k. Hence Σu converges by the assumption stated towards the end of § 80. If the terms of Σu are not all positive, but are numerically less than the corresponding (positive) terms of Σa, then Σu will be *absolutely* convergent.

(ii) The proof for divergence may be left to the student as it is so obvious.

This theorem is often called the **Comparison Test**, and is very useful.

8. $$1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2} + \dots \qquad (\alpha)$$

After the first, the terms are less than the corresponding terms of

$$1 + \frac{1}{1.2} + \frac{1}{2.3} + \dots + \frac{1}{(n-1)n} + \dots, \qquad (\beta)$$

that is, $$1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \dots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \dots$$

Thus (β) converges to 2, and therefore (α) converges.

9. $$1 + \frac{1}{2^k} + \frac{1}{3^k} + \frac{1}{4^k} + \dots \qquad (\gamma)$$

If $k > 2$, compare with series (α); (γ) will be thus seen to converge. If $k < 1$, compare with the harmonic series; (γ) will be seen to diverge. The series (γ) converges if only $k > 1$, though this proof does not show it.

10. $$\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \dots.$$

Comparison with the series (a), Ex. 8, shows that this series converges absolutely for every x. [See p. 149, (i).]

84. Maclaurin's Theorem. A function $f(x)$ can frequently be represented by an infinite series of ascending integral powers of x (a **Power Series** in x); we have had examples of such series for the functions $\sin x$ and $\cos x$ (p. 113), $\tan^{-1}x$ (p. 158) and $\log(1+x)$ (p. 165). We shall now give the general theorem of which these are particular cases; the way in which we derive it is not really a proof, but it *suggests* the theorem. For a more accurate statement and references, see § 87.

Suppose that $f(x)$ can be represented by a Power Series in x, say

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots + a_n x^n + \dots \quad \dots\dots(1)$$

We must assume the series to be convergent, say for $|x| \leqq c$; when the coefficients have been found in any particular case, the convergence can often be tested by the rules already given.

Now differentiate (1) once, twice, thrice, ...; then

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \dots + n a_n x^{n-1} + \dots \quad \dots\dots(2)$$

$$f''(x) = 1 \cdot 2a_2 + 2 \cdot 3a_3 x + \dots + (n-1) n a_n x^{n-2} + \dots \quad \dots\dots(3)$$

$$f'''(x) = 1 \cdot 2 \cdot 3a_3 + 2 \cdot 3 \cdot 4a_4 x + \dots + (n-2)(n-1) n a_n x^{n-3} + \dots \quad \dots\dots(4)$$

. . = ...

$$f^{(n)}(x) = 1 \cdot 2 \cdot 3 \dots (n-1) n a_n + 2 \cdot 3 \dots n(n+1) a_{n+1} x + \dots \quad \dots\dots(5)$$

. . = ...

Put $x=0$ in each of the equations (1), (2), ...; we thus find

$$f(0) = a_0; \quad f'(0) = a_1; \quad f''(0) = 1 \cdot 2a_2;$$

$$f'''(0) = 1 \cdot 2 \cdot 3a_3; \ \dots \ f^{(n)}(0) = 1 \cdot 2 \cdot 3 \dots n a_n; \dots.$$

Solving these for $a_0, a_1, a_2, \dots$, we obtain

$$a_0 = f(0); \quad a_1 = f'(0); \quad a_2 = \frac{1}{2!} f''(0);$$

$$a_3 = \frac{1}{3!} f'''(0); \ \dots \ a_n = \frac{1}{n!} f^{(n)}(0); \dots.$$

Thus, finally,

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\ldots+\frac{x^n}{n!}f^{(n)}(0)+\ldots\ldots\ldots(\text{M})$$

Equation (M) gives Maclaurin's Theorem for the representation (or expansion) of $f(x)$ as a Power Series in x.

If, instead of assuming a Power Series in x, we assume one in $(x-a)$, say

$$f(x)=b_0+b_1(x-a)+b_2(x-a)^2+\ldots+b_n(x-a)^n+\ldots,\ (1')$$

we differentiate as before, but in (1′) and in the equations formed by differentiating it we now put a instead of 0 for x; we thus find

$$f(x)=f(a)+(x-a)f'(a)$$
$$+\frac{(x-a)^2}{2!}f''(a)+\ldots+\frac{(x-a)^n}{n!}f^{(n)}(a)+\ldots\ \ldots\ldots(\text{T})$$

A useful form of equation (T) is obtained by putting $a+h$ for x; thus

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\ldots+\frac{h^n}{n!}f^{(n)}(a)+\ldots.\ (\text{T}_1)$$

Equation (T) gives Taylor's Theorem for the expansion of $f(x)$.

It must be specially noted that the expansions only hold so long as the series are convergent; if the series in (M) or (T) diverges, say for $x=c$, then the series is not equal to $f(c)$.

85. Some Standard Series.

1. **sin x.**

$$f(x)=\sin x \text{ and (p. 108) } f^{(n)}(x)=\sin\left(x+\frac{n\pi}{2}\right).$$

Hence $f(0)=0$; $f'(0)=\sin\frac{\pi}{2}=1$; $f''(0)=\sin\pi=0$;

$$f'''(0)=\sin\frac{3\pi}{2}=-1;\quad f^{\text{iv}}(0)=\sin 2\pi=0;\ \ldots.$$

Thus the coefficients of the even powers are zero; the value of $f^{(n)}(0)$ when n is odd is $+1$ and -1 alternately. Therefore

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots.$$

The series converges for every value of x (§ 83, Ex. 4).

2. **cos x.**

Here $f(x)=\cos x$ and $f^{(n)}(x)=\cos\left(x+\frac{n\pi}{2}\right)$. As in example 1, we find

$$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\dots,$$

the series converging for every x (§ 83, Ex. 5).

The series in examples 1 and 2 are the same as in § 44.

3. **$(1+x)^m$.**

Here $f(x)=(1+x)^m$ and $f^{(n)}(x)=m(m-1)\dots(m-n+1)(1+x)^{m-n}$, so that

$$f(0)=1, \quad f^{(n)}(0)=m(m-1)\dots(m-n+1),$$

and therefore

$$(1+x)^m = 1+mx+\frac{m(m-1)}{2!}x^2+\dots+\frac{m(m-1)\dots(m-n+1)}{n!}x^n+\dots.$$

The series converges absolutely if $|x|<1$ (§ 83, Ex. 1).

If a and b are not equal, $(a+b)^m$ can be written $a^m(1+b/a)^m$ or $b^m(1+a/b)^m$ and x put for b/a or a/b according as $|b|$ is less or greater than $|a|$.

This is the **Binomial Theorem**; m may be positive or negative, integral or fractional. Of course if m is a positive integer the series terminates with x^m.

The series for e^x, $\log(1+x)$ and $\tan^{-1}x$ have been already given; it will be a good exercise to establish them by Maclaurin's Theorem. The method of integrating a series, applied in the case of $\log(1+x)$ and of $\tan^{-1}x$, is very useful. It is often difficult to calculate $f^{(n)}(x)$, but without $f^{(n)}(0)$ Maclaurin's series cannot be found. Sometimes only a few terms of an expansion are required and the necessary derivatives can be calculated, often with a good deal of labour. See Ex. 4 in Exercises XXIII. (See also Ex. 2 in the same set for expansions consisting of a few terms.) The following example illustrates the method of integrating a series.

4. **$\sin^{-1}x$.** We know that $\sin^{-1}x$ is the integral of $1/\surd(1-x^2)$. Now, by the Binomial Theorem, since $1/\surd(1-x^2)=(1-x^2)^{-\frac{1}{2}}$,

$$\frac{1}{\surd(1-x^2)} = 1+\frac{1}{2}x^2+\frac{1.3}{2.4}x^4+\frac{1.3.5}{2.4.6}x^6+\dots.$$

Integrate, noting that the constant of integration is zero, since $\sin^{-1}x$ is zero when x is zero; then

$$\sin^{-1}x = x+\frac{1}{2}\frac{x^3}{3}+\frac{1.3}{2.4}\frac{x^5}{5}+\frac{1.3.5}{2.4.6}\frac{x^7}{7}+\dots,$$

which holds if $|x|<1$ (§ 83, Ex. 6).

5. An approximate value of a root of the equation $f(x)=0$ is known; show how to find a closer approximation to this root. (See p. 46, Ex. 26.)

Let a be the given approximation; then $f(a)$ is small but not zero. Choose a number b nearly equal to a but *such that $f(b)$ and $f(a)$ have opposite signs*. The graph of $f(x)$ will cross the x-axis between the points A and B whose abscissae are a and b, and therefore the root lies between a and b. Suppose $a+h$ to be the root; then $f(a+h)=0$. But by Taylor's Theorem [§ 84, (T_1)],

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2}f''(a)+\dots,$$

so that
$$0=f(a)+hf'(a)+\frac{h^2}{2}f''(a)+\dots.$$

Since h is small, we may, for the next approximation, neglect the square and higher powers of h; we thus find

$$0=f(a)+hf'(a) \quad \text{or} \quad h=-f(a)/f'(a),$$

and the new value for the root is $a+h$ or $a-f(a)/f'(a)$.

Similarly, if $b+k$ were the root, we should find

$$k=-f(b)/f'(b); \quad b+k=b-f(b)/f'(b).$$

We thus have two approximations, but we do not know which is the better; in fact one of them will often be worse than that from which we start. The rule for deciding between them is as follows:

If $f(a)$ and $f''(a)$ have the same sign, we take $a+h$; but if they have opposite signs, $f(b)$ and $f''(b)$ will have the same sign, and we take $b+k$.

It is necessary for the validity of the rule that $f'(x)$ and $f''(x)$ should neither vanish nor change sign from $x=a$ to $x=b$; for a proof of the rule see the author's larger treatise on the *Calculus*, p. 245. The method is known as **Newton's Rule.**

6. Solve the equation $\quad x+\sin x-\dfrac{\pi}{3}=0.$

Here $\;f(x)=x+\sin x-\dfrac{\pi}{3}\,;\quad f'(x)=1+\cos x\,;\quad f''(x)=-\sin x.$

By trial, the angle will be found to lie between 30° and 31° or, in radians since x is in radians, between 0·5236 and 0·5411. Now

$$f(0{\cdot}5236)=-0{\cdot}0236, \quad f'(0{\cdot}5236)=1{\cdot}8660;$$
$$f(0{\cdot}5411)=+0{\cdot}0089, \quad f'(0{\cdot}5411)=1{\cdot}8572.$$

Since $f''(x)$ is negative and $f(0{\cdot}5236)$ also negative, we take $a=0{\cdot}5236$, and for $a+h$, we find

$$a-\frac{f(a)}{f'(a)}=0{\cdot}5236+\frac{0{\cdot}0236}{1{\cdot}8660}=0{\cdot}5236+0{\cdot}0126,$$

so that $a+h=0{\cdot}5362$.

If we call this value a_1, we get $a_1-f(a_1)/f'(a_1)$ for a still better approximation; this new value is 0·5362674, the error in which is less than a unit of the last figure. In degrees, the angle is 30° 43′ 33″·0.

The above equation gives the solution of the problem: A is a point on the circumference of a circle and AB, AC are two chords which

trisect the area of the circle; find the angle ($\frac{1}{2}x$) between AB and the diameter through A.

86. Exponential Values of Sine and Cosine. We suppose that the student knows that complex numbers $a+ib$ may be combined by the ordinary rules of algebra. The symbols a and b denote real numbers (such as $2, -\frac{1}{3}, \sqrt{5}, \pi, \ldots$); the symbol i is "the imaginary unit," and is such that $i^2=-1$. Every integral power of i has one of the four values $i, -1, -i, +1$. Thus

$$i=i,\quad i^2=-1,\quad i^3=i\,.\,i^2=-i,\quad i^4=i\,.\,i^3=-i^2=+1;$$
$$i^5=i\,.\,i^4=i,\quad i^6=i\,.\,i^5=i^2=-1,\quad i^7=i\,.\,i^6=-i,$$
$$i^8=i\,.\,i^7=-i^2=+1;$$

and so on, the values $i, -1, -i, +1$ recurring periodically.

Similarly,

$$i^{-1}=1/i=i/i^2=-i;\qquad i^{-2}=1/i^2=-1;$$
$$i^{-3}=1/i^3=1/(-i)=+i;\qquad i^{-4}=1/i^4=+1;$$

and so on.

Now, in the series

$$1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\frac{x^6}{6!}+\frac{x^7}{7!}+\cdots,$$

put $i\theta$ for x, and it becomes

$$1+i\theta-\frac{\theta^2}{2!}-i\frac{\theta^3}{3!}+\frac{\theta^4}{4!}+i\frac{\theta^5}{5!}-\frac{\theta^6}{6!}-i\frac{\theta^7}{7!}+\cdots,$$

or, taking the real and the pure imaginary terms as separate series,

$$1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\frac{\theta^6}{6!}+\cdots+i\left(\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\frac{\theta^7}{7!}+\cdots\right),$$

that is, $\cos\theta+i\sin\theta.$

We may therefore take the symbol $e^{i\theta}$ to mean $\cos\theta+i\sin\theta$, and in the same way $e^{-i\theta}$ may be taken to mean $\cos\theta-i\sin\theta$. Hence we have

$$\cos\theta+i\sin\theta=e^{i\theta},\quad \cos\theta-i\sin\theta=e^{-i\theta},$$

and therefore, solving for $\cos\theta$ and $\sin\theta$,

$$\cos\theta=\frac{e^{i\theta}+e^{-i\theta}}{2},\quad \sin\theta=\frac{e^{i\theta}-e^{-i\theta}}{2i}.$$

These are called the Exponential Values of $\cos\theta$ and $\sin\theta$.

If we assume that the power $e^{i\theta}$ is subject to the ordinary laws of indices, it is easy to verify from these expressions that, for example,

$$\cos^2\theta + \sin^2\theta = 1, \quad \cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi.$$

Again, if we differentiate, assuming that the symbol i may be treated as a constant, we see that

$$\frac{d.\sin\theta}{d\theta} = \frac{1}{2i}\,\frac{d(e^{i\theta} - e^{-i\theta})}{d\theta} = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \cos\theta,$$

$$\frac{d.\cos\theta}{d\theta} = \frac{1}{2}\,\frac{d(e^{i\theta} + e^{-i\theta})}{d\theta} = \frac{i}{2}(e^{i\theta} - e^{-i\theta}) = -\sin\theta,$$

because $$\frac{i}{2} = \frac{i^2}{2i} = \frac{-1}{2i}.$$

Example 1. Express $Ae^{2xi} + Be^{-2xi}$ in terms of sine and cosine.

The expression $= A(\cos 2x + i\sin 2x) + B(\cos 2x - i\sin 2x)$

$= (A+B)\cos 2x + i(A-B)\sin 2x.$

Example 2. Express $Ae^{(3+2i)x} + Be^{(3-2i)x}$ in terms of a real exponential and of sine and cosine.

We have $\qquad e^{(3+2i)x} = e^{3x}.e^{2xi}; \quad e^{(3-2i)x} = e^{3x}.e^{-2xi}.$

Hence the given expression is equal to

$$e^{3x}(Ae^{2xi} + Be^{-2xi}) = e^{3x}\{(A+B)\cos 2x + i(A-B)\sin 2x\}.$$

Similarly

$$Ae^{(-a+ni)x} + Be^{(-g-ni)x} = e^{-ax}\{(A+B)\cos nx + i(A-B)\sin nx\}.$$

The exponential values of sine and cosine will be found useful a little later (§ 91).

87. Taylor's Theorem with Remainder. We merely state the Theorem for purposes of reference; a full discussion will be found in the author's larger treatise, Chap. XVIII.

If $f(x)$ and its first n derivatives are continuous (§ 10) from $x = a$ to the value of x considered, then

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \dots$$
$$+ \frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + R_n, \quad \dots(\text{T})$$

where $$R_n = \frac{(x-a)^n}{n!}f^{(n)}(x_1).$$

R_n is called the Remainder after n terms. All that is known of the number x_1 is that it lies between a and x; that is, $a<x_1<x$ if $x>a$, but $a>x_1>x$ if $x<a$.

When n tends to infinity the sum becomes an infinite series; if R_n tends to 0 the series will then converge and represent $f(x)$. If R_n does not tend to 0 the equation (T) is still true as it stands.

Maclaurin's Theorem is obtained by making $a=0$.

The following particular cases of (T) are useful. Let $x=a+h$; then

$$\text{(i)}\quad n=1;\quad f(a+h)=f(a)+hf'(x_1).$$

$$\text{(ii)}\quad n=2;\quad f(a+h)=f(a)+hf'(a)+\frac{h^2}{2}f''(x_2).$$

x_1 and x_2 may be different, but both lie between a and $a+h$.

EXERCISES. XXIII.

1. Prove that the following expansions hold for every x:

$$\text{(i)}\quad \sin(x+\alpha)=\sin\alpha+x\cos\alpha-\frac{x^2}{2!}\sin\alpha-\frac{x^3}{3!}\cos\alpha+\dots;$$

$$\text{(ii)}\quad \cos(x+\alpha)=\cos\alpha-x\sin\alpha-\frac{x^2}{2!}\cos\alpha+\frac{x^3}{3!}\sin\alpha+\dots;$$

$$\text{(iii)}\quad \cos^2 x=1-x^2+\frac{2^3x^4}{4!}-\frac{2^5x^6}{6!}+\dots;$$

$$\text{(iv)}\quad e^{x\cos\alpha}\cos(x\sin\alpha)=1+x\cos\alpha+\frac{x^2}{2!}\cos 2\alpha+\frac{x^3}{3!}\cos 3\alpha+\dots.$$

Show that $D^n e^{x\cos\alpha}\cos(x\sin\alpha)=e^{x\cos\alpha}\cos(x\sin\alpha+n\alpha)$.

2. By substituting the series for $\cos x$ and $\sin x$ and dividing, show that up to the terms stated the following approximations may be obtained:

$$\text{(i)}\quad \sec x=1+\frac{x^2}{2}+\frac{5x^4}{24}+\frac{61x^6}{720};$$

$$\text{(ii)}\quad \tan x=x+\frac{x^3}{3}+\frac{2x^5}{15}+\frac{17x^7}{315};$$

$$\text{(iii)}\quad x\cot x=1-\frac{x^2}{3}-\frac{x^4}{45}-\frac{2x^6}{945}.$$

Why can $\cot x$ not be expanded by Maclaurin's Theorem?

3. Show that if $|x|<1$,

$$\log(1+x+x^2)=x+\frac{x^2}{2}-\frac{2x^3}{3}+\frac{x^4}{4}+\dots.$$

4. Show that up to the terms given,

$$\text{(i)} \log(1+\sin x)=x-\frac{x^2}{2}+\frac{x^3}{6}-\frac{x^4}{12};$$

$$\text{(ii)} \log(1+\cos x)=\log 2-\frac{x^2}{4}-\frac{x^4}{96}.$$

Calculate only the first four derivatives in each case.

5. Show from the series for $\tan^{-1}x$ that the angle $\tan^{-1}\frac{1}{10}=5^\circ\ 42'\ 38''$.

***6.** Prove that the series for $\log(1+x)$ and $x(6+x)/(6+4x)$ are identical up to the term in x^3 (inclusive).

7. Find an expression for the error R in the approximation to $\log(1+x)$ given in example 6, namely

$$\log(1+x)=\frac{x(6+x)}{6+4x}+R.$$

Multiply by $6+4x$ and expand $\log(1+x)$; then

$$(6+4x)R=(6+4x)\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots\right)-x(6+x)$$

$$=-\frac{x^4}{6}+\frac{x^5}{5}-\ldots+(-1)^{n-1}\frac{2(n-3)x^n}{n(n-1)}+\ldots,$$

and therefore

$$-(3+2x)R=\frac{1}{3\,.\,4}x^4-\frac{2}{4\,.\,5}x^5+\ldots+(-1)^n\frac{n-3}{n(n-1)}x^n-\ldots$$

$$<\frac{x^4}{4}+\frac{x^5}{4}+\ldots+\frac{x^n}{4}+\ldots \text{ if } 0<x<1,$$

so that

$$-R<\frac{x^4}{4(1-x)(3+2x)} \text{ if } 0<x<1.$$

8. Assuming the equation $\log(1+x)=\dfrac{x(a+bx)}{1+cx+dx^2}+R$, find a, b, c, d so that the first four terms in the expansion of the fraction may be the same as those in the series for $\log(1+x)$, and investigate an expression for R.

The simplest method is to write the equation in the form

$$(1+cx+dx^2)R=(1+cx+dx^2)\log(1+x)-x(a+bx);$$

then expand $\log(1+x)$ and equate to zero the coefficients of x, x^2, x^3 and x^4 on the right side. This gives $a=1$, $b=\frac{1}{2}$, $c=1$, $d=\frac{1}{6}$. We then have

$$(6+6x+x^2)R=\frac{1\,.\,2}{3\,.\,4\,.\,5}x^5-\frac{2\,.\,3}{4\,.\,5\,.\,6}x^6+\frac{3\,.\,4}{5\,.\,6\,.\,7}x^7-\ldots$$

$$<\frac{1}{5}\frac{x^5}{1-x} \text{ if } 0<x<1.$$

* **Examples 6-13 are taken from Schlömilch's *Übungsbuch*, Vol. I.**

Hence $$\log(1+x)=\frac{x(6+3x)}{6+6x+x^2}+\frac{\rho x^5}{(6+6x+x^2)(1-x)},$$
where $$0<x<1 \quad\text{and}\quad 0<\rho<\tfrac{1}{5}.$$
If $0<x<\frac{2}{3}$, then $R=\rho x^5$, where $0<\rho<\frac{1}{180}$.

9. Show that $0{\cdot}2623574<\log 1{\cdot}3<0{\cdot}2623709$.

10. Prove that $$\cos x=\frac{12-5x^2}{12+x^2}+\rho x^6,$$
where $$0<\rho<\frac{1}{480} \text{ if } 0<x<\frac{3\pi}{2}.$$

11. Prove that $$\sin x=\frac{x(60-7x^2)}{60+3x^2}+\rho x^7,$$
where $$0<\rho<\frac{1}{4200} \text{ if } 0<x<\frac{3\pi}{2}.$$

12. Show that the equation $\log(1+x)=\frac{3}{4}x$ has a root between 0 and 1, and from the approximate value of $\log(1+x)$ in example 8 show that $\sqrt{3}-1$ is an approximation to the root. Deduce the more accurate approximation $0{\cdot}73360$, which is correct to 5 decimals.

13. Show by graphical considerations that the equation
$$(1+x^2)\cos x=1$$
has an infinite number of roots, and that one root lies between 0 and $\pi/2$; prove that for this root $x=\sqrt{(6/5)}$ approximately. This angle is $62^\circ\ 46'$.

14. Find the real root of $3x^3+5x-40=0$.

15. Solve the equation $x=\cos x$.

16. Find the root of the equation $\tan x=2x$ that lies between 0 and $\pi/2$.

17. Solve the equation $(1+x)^x=27{\cdot}34$.
Write in the form $x\log(1+x)-\log 27{\cdot}34=0$.

18. Prove that if $|x|<1$,
$$\log\{x+\sqrt{(1+x^2)}\}=x-\frac{1}{2}\frac{x^3}{3}+\frac{1.3}{2.4}\frac{x^5}{5}-\frac{1.3.5}{2.4.6}\frac{x^7}{7}+\dots.$$

19. If $y=2\log\left\{\dfrac{2}{1+\sqrt{(1-x)}}\right\}$, prove that, $(|x|<1)$,

(i) $$\frac{dy}{dx}=\frac{1}{x}\{(1-x)^{-\frac{1}{2}}-1\}=\frac{1}{2}+\frac{1.3}{2.4}x+\frac{1.3.5}{2.4.6}x^2+\dots;$$

(ii) $$y=\frac{1}{2}\frac{x}{1}+\frac{1.3}{2.4}\frac{x^2}{2}+\frac{1.3.5}{2.4.6}\frac{x^3}{3}+\dots.$$

20. The time of a complete oscillation of a simple pendulum of length l, oscillating through an angle α on each side of the vertical, is $4K\sqrt{(l/g)}$, where

$$K=\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{(1-k^2\sin^2\phi)}},\quad k=\sin\tfrac{1}{2}\alpha.$$

Show that $$K=\frac{\pi}{2}\left\{1+\left(\frac{1}{2}\right)^2k^2+\left(\frac{1\,.\,3}{2\,.\,4}\right)^2k^4+\left(\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\right)^2k^6+\dots\right\}.$$

88. Differential Equations. In §§ 67-69 various problems have led to differential equations; as such equations occur frequently in applications, we consider some simple types.

An **ordinary differential equation** is an equation between one independent variable, one dependent variable and one or more derivatives of the dependent variable. The **order** of the equation is that of the highest derivative in it; the **degree** is that of the highest derivative when the equation is cleared of fractions and the powers of the derivatives are positive integers.

For example, the equation

$$x^2\frac{d^2y}{dx^2}-2y=x$$

is of the 2nd order and the 1st degree. The equation

$$xy\left(\frac{dy}{dx}\right)^2+(x^2-y^2-a^2+b^2)\frac{dy}{dx}-xy=0$$

is of the 1st order and the 2nd degree. Equation (3), p. 185, is of the 2nd order and the 2nd degree.

Just as the complete integral of $3x^2$ is x^3+A, where A is any constant, so the complete integral of any differential equation contains one or more arbitrary constants, and *the number of such constants is equal to the order of the equation.* In order to see why we may expect this number of constants, let us consider the formation of a differential equation from an equation that may be regarded as its complete integral.

From the equation $y=Ax^2+Bx$ form two differential equations, (i) one from which the constant B is absent and (ii) one from which both constants A and B are absent.

Let accents denote x-derivatives, and differentiate the given equation once; we have

$$y=Ax^2+Bx\ \dots\dots\dots(1);\qquad y'=2Ax+B.\ \dots\dots\dots\dots\dots(2)$$

To eliminate one constant B we must have two equations; the required second equation is obtained by differentiation, and this process introduces y'. Solve (2) for B and put the value so found in (1); we thus obtain

$$xy' - y = Ax^2, \quad\text{.................................}(3)$$

an equation of the first order.

To eliminate the two constants A and B, we must have three equations; we have already (1) and (2), and the required third equation may be found by differentiating (2). We thus introduce the second derivative y'', and have the new equation

$$y'' = 2A. \quad\text{.......................................}(4)$$

We now eliminate A between (3) and (4), or A and B between (1), (2) and (4), and obtain

$$x^2y'' - 2xy' + 2y = 0, \quad\text{.............................}(5)$$

an equation of the second order.

If A is regarded as a *given* constant, the complete integral of (3) is equation (1), while, if both A and B are arbitrary, equation (1) is the complete integral of (5).

If we give particular values to A and B, we get **particular integrals** of (5). Thus $y = x^2 + x$, $y = -x^2$, $y = \frac{1}{2}x$ are particular integrals.

For other examples see p. 112, examples 56, 57; p. 172, example 4; and p. 174, examples 26, 28, 29.

From the consideration of the process by which a differential equation is derived from an equation that contains the two variables and one or more arbitrary constants, we have the rule for testing the complete integral or solution of a given differential equation:

From the supposed complete integral and the equations obtained by differentiating it, eliminate the arbitrary constants; if the resulting differential equation is the same as the given one the solution is correct.

89. Equations of the First Order and the First Degree. These equations are of the form

$$A\frac{dy}{dx} + B = 0,$$

where A and B are any functions of x and y (or constants). The following are the more important types:

Type I. Variables Separable. The variables are said to be separable when the equation can be written in the form

$$F(y)\frac{dy}{dx} = f(x) \quad\text{or}\quad F(y)\,dy = f(x)\,dx,$$

where $f(x)$ is a function of x alone and $F(y)$ a function of y alone. The complete solution is

$$\int F(y)\,dy = \int f(x)\,dx + \text{constant}.$$

Example 1. $$2(x+1)\frac{dy}{dx} - 3(y+2) = 0.$$

We have $$\frac{2dy}{y+2} = \frac{3dx}{x+1};$$

therefore $$2\log(y+2) = 3\log(x+1) + \text{const.}$$

or $$\log[(y+2)^2] = \log[(x+1)^3] + \text{const.}$$

It is convenient to write the constant in the form $\log C$; then the solution may be put in the simple form

$$(y+2)^2 = C(x+1)^3.$$

It is often useful to choose the constant in a particular form, as here; if $\log C$ is arbitrary so is C, and neither form is more general than the other. (*Test the solution by the rule given at the end of* § 88.)

Example 2. Find the curve in which the subtangent is double the abscissa, and which passes through the point $(a, 2a)$.

From the expression on p. 176, the subtangent at (x, y) is y/y', and therefore $y/y' = 2x$. Hence $2dy/y = dx/x$, and therefore

$$2\log y = \log x + \log C \quad \text{or} \quad y^2 = Cx.$$

The curve goes through $(a, 2a)$, and therefore $4a^2 = Ca$ or $C = 4a$.

The required curve is therefore the parabola $y^2 = 4ax$.

Type II. Homogeneous Equations. An equation is called homogeneous when it is of the form $Ay' + B = 0$, where A and B are homogeneous and of the same degree in x and y.

To solve, change the dependent variable from y to v by the substitution $y = vx$. This gives $y' = v'x + v$. For y and y' put their values in terms of v and v'; the variables will then be separable.

Example 3. $$(xy + x^2)y' + y^2 = 0.$$

Putting vx for y and $v'x + v$ for y', we get

$$x^2(v+1)(v'x+v) + v^2x^2 = 0,$$

or, dividing by x^2 and rearranging,

$$\left(\frac{1}{v} - \frac{1}{2v+1}\right)dv + \frac{dx}{x} = 0.$$

The integral is $\log v - \frac{1}{2}\log(2v+1) + \log x = \text{const.},$
or $\log v^2 + \log x^2 = \log(2v+1) + \log C,$
or $xy^2 = C(2y+x)$
when y/x is put for v. (*Test the solution.*)

Type III. Linear Equations. The equation is said to be linear when it is of the form $Ay' + By + C = 0$, *where* A, B, C *are functions of* x *alone or constants.*

Divide by A and put P for B/A and Q for $-C/A$; the equation becomes

$$\frac{dy}{dx} + Py = Q.$$

First suppose $Q = 0$; the variables are then separable, and we obtain

$$\frac{dy}{y} + P\,dx = 0 \quad \text{or} \quad \log y + P_1 = \text{const.},$$

where $P_1 = \int P\,dx$. We may write $P_1 = \log e^{P_1}$, and the solution becomes

$$\log(ye^{P_1}) = \text{const.} \quad \text{or} \quad ye^{P_1} = C.$$

The student should verify by differentiation that

$$\frac{d}{dx}(ye^{P_1}) = \left(\frac{dy}{dx} + Py\right)e^{P_1}.$$

Now take the general case. *Multiply by* e^{P_1}; the left side becomes

$$\left(\frac{dy}{dx} + Py\right)e^{P_1}, \quad \text{that is,} \quad \frac{d}{dx}(ye^{P_1}).$$

Therefore
$$\frac{d}{dx}(ye^{P_1}) = Qe^{P_1},$$

and the solution is $ye^{P_1} = \int Qe^{P_1}\,dx + \text{const.}$

Cor. The equation $y' + Py = Qy^n$ may be reduced to the linear form by putting $v = y^{-n+1}$, and taking v as new dependent variable.

Example 4. $(1+x^2)\dfrac{dy}{dx} + xy = x(1+x^2).$

Divide by $(1+x^2)$; then we find

$$P = \frac{x}{1+x^2}; \quad P_1 = \int\frac{x\,dx}{1+x^2} = \frac{1}{2}\log(1+x^2); \quad Q = x,$$

and therefore $e^{P_1} = \sqrt{(1+x^2)}$. Hence

$$\frac{d}{dx}\{y\sqrt{(1+x^2)}\} = x\sqrt{(1+x^2)},$$

so that $$y\sqrt{(1+x^2)} = \tfrac{1}{3}(1+x^2)^{\frac{3}{2}} + C.$$

Example 5. $$yy' + ay^2 = be^{-ax}.$$

Let $y^2 = v$ and the equation becomes linear, namely

$$\frac{dv}{dx} + 2av = 2be^{-ax},$$

of which the solution is

$$y^2 = v = Ce^{-2ax} + \frac{2b}{a}e^{-ax}.$$

For further examples in which the determination of the constant so as to satisfy particular conditions is illustrated, see the examples in § 69.

Remark. The student must observe carefully which letter denotes the independent variable; in the examples of § 69 the independent variable is t, and $P_1 = \int P\,dt$.

90. Linear Equations. A linear equation is one in which the dependent variable and its derivatives occur only in the first degree and the coefficients are functions of the independent variable alone (or constants). The typical linear equation of the second order is, when the coefficient of y'' is made unity by division if necessary,

$$y'' + Py' + Qy = R, \quad\quad(1)$$

where P, Q and R are functions of x alone.

The complete integral of all linear equations is the sum of two functions:

I. *The Complementary Function* (C.F), which is the complete integral of the equation when R (or in general the term independent of y and its derivatives) is zero. This function will contain two (when the equation is of the n^{th} order, n) arbitrary constants.

II. *The Particular Integral* (P.I.), which is any solution whatever of the equation as it stands. This function contains no arbitrary constant.

We prove the proposition for equation (1) of the second order, but the reasoning is quite general.

If $y=u$ and $y=v$ satisfy

$$y''+Py'+Qy=0, \qquad (2)$$

so does $y=Au+Bv$, where A, B are any constants. For if u and v satisfy (2), then

$$u''+Pu'+Qu=0, \quad v''+Pv'+Qv=0,$$

and

$$\frac{d^2(Au+Bv)}{dx^2}+P\frac{d(Au+Bv)}{dx}+Q(Au+Bv)$$
$$=A(u''+Pu'+Qu)+B(v''+Pv'+Qv)$$
$$=0,$$

and therefore $y=Au+Bv$ satisfies (2), and is the complete integral of (2) since it contains two arbitrary constants.

Next, if w is a value of y which satisfies (1) and if we put $Au+Bv+w$ for y, we have

$$y''+Py'+Qy=A(u''+Pu'+Qu)+B(v''+Pv'+Qv)+w''+Pw'+Qw$$
$$=R,$$

since u and v satisfy (2) and w satisfies (1). This value of y therefore satisfies (1), and since it contains two arbitrary constants it is the complete solution.

If the equation had been of the 3[rd] order there would have been three functions, u, v, z, and the solution would have been

$$y=Au+Bv+Cz+w,$$

and so on.

The only equations we consider are those in which P, Q are constants.

91. The Complementary Function. The equation for the complementary function is

$$y''+2ay'+by=0, \qquad (1)$$

where a, b are constants. (The notation for the coefficients has been changed to call attention to the fact that they are constants.)

I. Let $y=e^{\lambda x}$ (λ constant); then

$$y''+2ay'+by=(\lambda^2+2a\lambda+b)e^{\lambda x}.$$

If, therefore, λ is a root of the equation (the *auxiliary* equation)

$$\lambda^2+2a\lambda+b=0, \qquad (2)$$

$e^{\lambda x}$ will satisfy (1). If λ_1 and λ_2 are the two roots of (2), then $e^{\lambda_1 x}$ and $e^{\lambda_2 x}$ are two integrals of (1), and the complete solution of (1) is

$$y=Ae^{\lambda_1 x}+Be^{\lambda_2 x}.$$

We must consider the roots more carefully, however.

II. If $a^2 = b$, the two roots are equal and there is only one distinct integral, namely e^{-ax}, because now $\lambda_1 = \lambda_2 = -a$ and $(A+B)$ is really only one constant. To find a second integral, let $y = e^{-ax}u$; then

$$y' = e^{-ax}u' - ae^{-ax}u, \quad y'' = e^{-ax}u'' - 2ae^{-ax}u' + a^2e^{-ax}u,$$

and therefore, since $b = a^2$,

$$y'' + 2ay' + a^2y = e^{-ax}u''.$$

The equation (1), when $a^2 = b$, is therefore satisfied if $u'' = 0$, that is if $u = A + Bx$. Hence

$$y = (A + Bx)e^{-ax} = Ae^{-ax} + Bxe^{-ax}$$

satisfies (1), and, since it contains two arbitrary constants, is the complete solution.

We see that the second integral is xe^{-ax}; when $-a$ is a double root of the auxiliary equation e^{-ax} and xe^{-ax} are the two integrals for the C.F.

III. If $a^2 < b$, the two roots are imaginary. Let $a^2 - b = -n^2$, where n is real; the two roots are $-a + ni$ and $-a - ni$ $(i^2 = -1)$. The complete solution is

$$\begin{aligned} y &= Ae^{(-a+ni)x} + Be^{(-a-ni)x} \\ &= e^{-ax}\{(A+B)\cos nx + i(A-B)\sin nx\} \end{aligned}$$

by § 86, example 2. Let $A + B = A'$, $i(A - B) = B'$; since A and B may be any constants, A' and B' are also arbitrary. Hence the complete solution is

$$y = e^{-ax}(A' \cos nx + B' \sin nx).$$

Example 1. $y'' + y' - 6y = 0.$
Aux. Eq., $\lambda^2 + \lambda - 6 = 0$; $\lambda_1 = 2$, $\lambda_2 = -3$.
Solution, $y = Ae^{2x} + Be^{-3x}$.

Example 2. $y'' + 4y' + 13y = 0.$
Aux. Eq., $\lambda^2 + 4\lambda + 13 = 0$; $\lambda_1 = -2 + 3i$, $\lambda_2 = -2 - 3i$.
Solution, $y = e^{-2x}(A \cos 3x + B \sin 3x)$.

(It does not matter whether we call the constants A, B or A' B' in the *final* result.)

Example 3. $y^{\text{iv}} - y'' - 2y' + 2y = 0.$

The method is the same as for an equation of the 2nd order; put $y = e^{\lambda x}$ and proceed as before.

Aux. Eq.,

$$\lambda^4 - \lambda^2 - 2\lambda + 2 = 0\ ; \quad \lambda_1 = 1 = \lambda_2,\ \lambda_3 = -1 + i,\ \lambda_4 = -1 - i.$$

The integrals arising from the double root 1 are e^x and xe^x, which together contribute $Ae^x + Bxe^x$. The other two roots give

$$e^{-x}(A_1 \cos x + B_1 \sin x).$$

Hence the complete solution is

$$y = Ae^x + Bxe^x + e^{-x}(A_1 \cos x + B_1 \sin x),$$

which contains four constants as it should.

92. The Particular Integral. The most important cases are those in which R is a sum of terms of the form ce^{ax}, $c \sin ax$, $c \cos ax$, and the simplest method of finding a solution is by substitution.

If R is the sum of two or more terms, we may find the particular integral for each separately and then add the solutions to obtain the whole P.I. The equation for which the particular integral is to be found is, a, b being constant,

$$y'' + 2ay' + by = R. \quad \text{....................(1)}$$

I. $R = ce^{ax}$. Put $y = Le^{ax}$ and try to determine the constant L so that (1) may be satisfied. We find

$$L(a^2 + 2aa + b)e^{ax} = R = ce^{ax},$$

and Le^{ax} will satisfy if $L = c/(a^2 + 2aa + b)$.

But if a is a root of the auxiliary equation (and therefore Ae^{ax} a term of the C.F.), $a^2 + 2aa + b = 0$ and $L = c/0$, that is, L is not determined. In this case try Lxe^{ax} or Lx^2e^{ax} according as a is a single or a double root of the auxiliary equation.

Example 1. $y'' - 4y' + 4y = e^{-3x} + e^{2x}.$

Aux. Eq., $\lambda^2 - 4\lambda + 4 = 0$; $\lambda = 2$, twice,

and therefore C.F. $= Ae^{2x} + Bxe^{2x}$.

To find P.I. take e^{-3x} and e^{2x} separately, that is, find the P.I. for

$$y'' - 4y' + 4y = e^{-3x} \quad \text{.............................. (i)}$$

and

$$y'' - 4y' + 4y = e^{2x}, \quad \text{..............................(ii)}$$

and add the solutions to obtain the complete P.I. This course usually simplifies the work.

In (i) try $y=Le^{-3x}$; this gives $25L=1$ or $L=1/25$.

In (ii) try $y=Mx^2e^{2x}$, since 2 is a double root of the auxiliary equation; this gives $2M=1$ or $M=\frac{1}{2}$. Hence

$$\text{P.I.}=\tfrac{1}{25}e^{-3x}+\tfrac{1}{2}x^2e^{2x},$$

and

$$y=\text{C.F.}+\text{P.I.}$$
$$=Ae^{2x}+Bxe^{2x}+\tfrac{1}{25}e^{-3x}+\tfrac{1}{2}x^2e^{2x}.$$

II. $R=c\sin ax+c'\cos ax$.

To find P.I. try $y=L\sin ax+M\cos ax$; then

$$(-a^2L-2\alpha aM+bL)\sin ax+(-a^2M+2\alpha aL+bM)\cos ax$$
$$=R=c\sin ax+c'\cos ax,$$

and this equation will be an identity if

$$(b-a^2)L-2\alpha aM=c,\quad 2\alpha aL+(b-a^2)M=c'$$

or

$$L=\frac{(b-a^2)c+2\alpha ac'}{(b-a^2)^2+4\alpha^2a^2},\quad M=\frac{-2\alpha ac+(b-a^2)c'}{(b-a^2)^2+4\alpha^2a^2}.$$

This solution holds if $\alpha=0$, provided b is not equal to a^2: but if $\alpha=0$ and $b=a^2$ the values of L, M are not determined. In this case $\sin ax$ and $\cos ax$ belong to C.F.; the form to try is

$$y=x(L\sin ax+M\cos ax),$$

and it will be found that $L=c'/2a$, $M=-c/2a$.

For a method of solving when R is of the form $e^{\beta x}\sin ax$ (or $e^{\beta x}\cos ax$), see examples 41–44 in *Exercises XXIV.*

It may be noted that by expressing the sines and cosines as exponentials the type II. reduces to type I.; the student should try to solve some examples in this way.

Example 2. $\ddot{x}+k\dot{x}+\mu x=c\sin(pt-\alpha)$.

The equation occurs frequently in dynamical and electrical theory. The independent variable is t (see p. 49), and this fact must be carefully noted.

The C.F. is given on pages 172, 174; the cases for $\mu-\frac{1}{4}k^2$ positive and negative must be distinguished. For P.I. try

$$x=L\sin(pt-\alpha)+M\cos(pt-\alpha).$$

By substitution, we find

$$(-p^2L-kpM+\mu L)\sin(pt-\alpha)$$
$$+(-p^2M+kpL+\mu M)\cos(pt-\alpha)=c\sin(pt-\alpha),$$

and this equation will be an identity if

$$(\mu-p^2)L-kpM=c,\quad kpL+(\mu-p^2)M=0,$$

whence $$L=\frac{(\mu-p^2)c}{(\mu-p^2)^2+k^2p^2},\quad M=\frac{-kpc}{(\mu-p^2)^2+k^2p^2},$$

and $$\text{P.I.}=c\{(\mu-p^2)\sin(pt-a)-kp\cos(pt-a)\}/D$$
$$=c\sin(pt-a-a_1)/\sqrt{D},$$

where $$D=(\mu-p^2)^2+k^2p^2 \quad\text{and}\quad \tan a_1=kp/(\mu-p^2).$$

If $\mu-\frac{1}{4}k^2=n^2$, a positive number, the C.F. is (p. 172) $Ae^{-\frac{1}{2}kt}\cos(nt+B)$, and therefore, k being positive, tends to zero when kt becomes large, so that the complete integral becomes practically identical with the P.I.

If $k=0$ and $\mu=p^2$, $\text{P.I.}=-\dfrac{c}{2p}t\cos(pt-a)$.

III. R = polynomial in x. Try a polynomial in x of the same degree; if the auxiliary equation has one or more zero roots, a polynomial of higher degree will be necessary.

Example 3. $$y''+2y'=9x^2+5x+1.$$

The auxiliary equation has 0 and -2 for roots, so that
$$\text{C.F.}=A+Be^{-2x}.$$

For P.I. try $$y=Lx^3+Mx^2+Nx+P,$$

then $$6Lx^2+(6L+4M)x+(2M+2N)=9x^2+5x+1.$$

Equating coefficients, we find $L=\frac{3}{2}$, $M=-1$, $N=\frac{3}{2}$; but the constant P is not determined. The value of P, however, is really arbitrary, and may be absorbed in the constant term A of the C.F. Hence
$$y=A+Be^{-2x}+\tfrac{3}{2}x^3-x^2+\tfrac{3}{2}x.$$

93. Two Special Forms. The following equations of the second order
$$\ddot{x}=f(\dot{x})\;\ldots\ldots\ldots(1),\qquad \ddot{x}=f(x)\;\ldots\ldots\ldots\ldots(2)$$

are of importance in Dynamics. In (1) $f(\dot{x})$ is a function of $\dot{x}$, the first derivative of x; in (2) $f(x)$ is a function of the *dependent* variable x.

To integrate, let $\dot{x}=v$ in both cases and reduce each to an equation of the first order; the reduction is effected in different ways in the two cases, though the second method may also be used in case (1).

In case (1), $$\ddot{x}=\frac{d\dot{x}}{dt}=\frac{dv}{dt},$$

and the new equation is $$\frac{dv}{dt}=f(v).\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

In case (2), $$\ddot{x}=\frac{dv}{dt}=\frac{d(\frac{1}{2}v^2)}{dx} \quad \text{(see p. 51, ex. 2)},$$

and the new equation is $$\frac{d(\frac{1}{2}v^2)}{dx}=f(x). \qquad \text{.......................(4)}$$

In both cases the variables are separable, and it may be possible to find v. Suppose such an equation to have been obtained (it is called a *first integral*); put $\dot{x}$ for v, and we have an equation of the first order which it may be possible to integrate.

For example, let $\ddot{x}=-n^2x$; then (4) gives

$$\tfrac{1}{2}v^2=\int -n^2x\,dx+C=-\tfrac{1}{2}n^2x^2+\tfrac{1}{2}n^2a^2$$

if $C=\frac{1}{2}n^2a^2$. Solve for v; then

$$\frac{dx}{dt}=v=n\sqrt{(a^2-x^2)}, \quad n\,dt=dx/\sqrt{(a^2-x^2)}.$$

Integrate: $\quad nt=\sin^{-1}\dfrac{x}{a}+b \quad$ or $\quad x=a\sin(nt-b).$

The two arbitrary constants are a and b.

For examples, see *Exercises XXIV.*, 51–55; the example just worked can of course be solved more simply by the methods of § 91, but it shows very clearly how to proceed in the general case in which $f(x)$ is not so simple.

EXERCISES. XXIV.

The student should note that, just as in ordinary integration, a change of variable will frequently reduce a differential equation to a standard type.

Integrate equations 1–20.

1. $x\dfrac{dy}{dx}+y^2=1.$

2. $x^2\dfrac{dy}{dx}+y=x\dfrac{dy}{dx}.$

3. $(1+x^2)(1+y^2)\dfrac{dy}{dx}+2xy(1-y^2)=0.$

4. $(u-v)^2\dfrac{du}{dv}=1.$

5. $(y-x)\dfrac{dy}{dx}+y=0.$

6. $(x^2+y^2)\dfrac{dy}{dx}+2xy=0.$

7. $x\dfrac{dy}{dx}-y=\sqrt{(x^2+y^2)}.$

8. $(x+y^2)\dfrac{dy}{dx}=y.$

9. $y\left(\dfrac{dy}{dx}\right)^2+2x\dfrac{dy}{dx}=y.$

10. $\dfrac{dy}{dx}+y\cot x=\cos x.$

11. $(1+x^2)\dfrac{dy}{dx}=xy+1+x^2.$

12. $(1-x^2)\dfrac{dy}{dx}+xy=ax.$

13. $\dfrac{dy}{dx}+y\tan x=\sec x.$

14. $r\dfrac{dp}{dr}+r+p=0.$

15. $\dfrac{dx}{dt}+2xt=1+2t^2.$

16. $xy(1-x^2)\dfrac{dy}{dx}+x^2=1+x^2y^2.$

17. $2xy\dfrac{dy}{dx}+x^2-y^2=a^2.$

18. $(1-x^2)\dfrac{dy}{dx}=xy(1+ay).$

19. $\dfrac{dy}{dx}+y=y^2\sin x.$

20. $\cos t\dfrac{dx}{dt}+x\sin t=x^2.$

21. The normal at the point $P(x, y)$ on a curve meets the x-axis at G; if $OG=k\,.\,OP$, O being the origin, show that the differential equation of the curve is

$$x+y\frac{dy}{dx}=k\sqrt{(x^2+y^2)},$$

and find the Cartesian equation.

22. A curve crosses the y-axis at B, and A is the projection on the x-axis of a point $P(x, y)$ on the curve; if the area $OAPB$ is equal to $x\sqrt{(ay)}$, show that the differential equation of the curve is

$$x\frac{dy}{dx}+2y=2y\sqrt{\left(\frac{y}{a}\right)},$$

and find the Cartesian equation.

23. The arc s of a curve from the point $(0, a)$ to the point (x, y) it is given by the equation $y^2=s^2+a^2$; find the equation of the cu[illegible]

In examples 24–40, accents denote x-derivatives and points t-de[illegible]tives; find the complete integrals of the equations.

24. $y''+ay'-2a^2y=0.$

25. $\ddot{x}+(a+b)\dot{x}+abx=0.$

26. $2y''+3y'-2y=e^{3x}.$

27. $6\ddot{x}+5\dot{x}+x=e^{-t}.$

28. $y'''-2y'-4y=e^{x}.$

29. $y''+2y'+5y=\cos 2x.$

30. $y''+6y'+18y=90\cos 3x.$

31. $y''+9y=a\sin^2 x.$

32. $y''+9y=a\sin 3x.$

33. $y''+y=a\sin x\sin$ [illegible]

34. $y''+y'+y=\sin x.$

35. $\ddot{x}+6\dot{x}+13x=a\,c$[illegible]$2t.$

36. $\ddot{x}+2\dot{x}=e^{-2t}+t.$

37. $y'''-3y''+4y=$ [illegible]$^{x}+be^{-x}.$

38. $y^{\text{iv}}-2y''+y=(x+1)^2.$

39. $3y''-y'-4y=$ [illegible]

40. $\ddot{x}+a\dot{x}-2a^2x=bt+c.$

Find the particular integrals of equations 41–4[illegible] by assuming for y the forms suggested:

41. $y''+4y'+5y=e^x\sin 3x$; $\quad y=e^x(L\sin 3x+M\cos 3x).$

42. $y'' + n^2 y = ae^{-mx} \sin nx$; $y = e^{-mx}(L \sin nx + M \cos nx)$.

43. $y'' + 6y' + 8y = ae^x x^2$; $y = e^x(Lx^2 + Mx + N)$.

44. $y'' + 2y' + 2y = e^{-x} \sin x$; $y = xe^{-x}(L \sin x + M \cos x)$.

Find the particular integrals of equations 45, 46 (i) when n^2 and p^2 are unequal, (ii) when $n^2 = p^2$.

45. $\ddot{x} + n^2 x = a \sin pt$. **46.** $\ddot{x} + n^2 x = ae^{-pt} \sin nt$.

47. Find the integral of the equation $a^2\ddot{x} + 2a\dot{x} + x = 0$ which satisfies the conditions that $x = b$ and $\dot{x} = 0$ when $t = 0$. If a and b are positive, sketch the integral curve.

48. Determine the constants in the integral of the equation

$$\ddot{x} + 4\dot{x} + 5x = a(\cos t - \sin t),$$

so that $x = 0$ and $\dot{x} = 0$ when $t = 0$. State the value to which x tends when t becomes large.

49. Determine the constants in the integral of the equation

$$\ddot{x} + a^2 x = \sin t$$

(a not equal to unity), so that $x = 0$ and $\dot{x} = 0$ when $t = 0$.

If $a = 1 + h$, what does the solution become when h tends to zero?

50. If a is positive, what does the integral of the equation

$$\ddot{x} + 2a\dot{x} + (a^2 + b^2)x = E \sin pt$$

tend to for large values of t?

51. Integrate the equation $\ddot{x} + k\dot{x}^2 = g$ (k and g positive), choosing the constants so that $x = 0$ and $\dot{x} = 0$ when $t = 0$. To what value does $\dot{x}$ tend when t tends to infinity?

52. Integrate the equation $\ddot{x} + k\dot{x}^2 = -g$ (k and g positive), choosing the constants so that $x = 0$, $\dot{x} = V$ when $t = 0$.

53. Find a first integral of the equation of motion of a simple pendulum at length l, namely,

$$l\ddot{\theta} + g \sin \theta = 0,$$

choosing the constant so that $\dot{\theta} = 0$ when $\theta = \alpha$.

54. Integrate $\dfrac{d^2 V}{dr^2} + \dfrac{n}{r}\dfrac{dV}{dr} = 0$.

55. Find a first integral of the equation $\ddot{x} = -\mu/x^2$ (μ positive), given that $\dot{x} = 0$ when $x = a$.

NOTE ON PARTIAL DIFFERENTIATION.

A variable may depend for its value on two or more variables that are themselves independent, that is, are such that one of them may vary while the others remain constant. Thus the area, u say, of a triangle in terms of two sides x, y and the included angle z is $\frac{1}{2}xy\sin z$; u is a function of the three independent variables x, y, z.

$$\text{If } x \text{ alone varies,} \quad \frac{du}{dx}=\tfrac{1}{2}y\sin z;$$

$$\text{if } y \text{ alone varies} \quad \frac{du}{dy}=\tfrac{1}{2}x\sin z;$$

$$\text{if } z \text{ alone varies} \quad \frac{du}{dz}=\tfrac{1}{2}xy\cos z.$$

These three derivatives are called *partial derivatives* or *partial differential coefficients* of u with respect to x, y and z respectively. To call attention to the fact that in differentiating with respect to one variable the other independent variables are constant, it is now customary to use the symbol ∂ instead of d; thus, when $u=\frac{1}{2}xy\sin z$,

$$\frac{\partial u}{\partial x}=\tfrac{1}{2}y\sin z, \qquad \frac{\partial u}{\partial y}=\tfrac{1}{2}x\sin z, \qquad \frac{\partial u}{\partial z}=\tfrac{1}{2}xy\cos z.$$

A function u of two variables x and y is denoted by $u=f(x, y)$ or $u=F(x, y)$, the letters x and y being separated by a comma. Similarly $f(x, y, z)$, $\phi(x, y, z, t)$ denote functions of x, y, z and x, y, z, t.

The equation $z=f(x, y)$, when x, y, z are the coordinates of a point in space referred to three rectangular axes, represents a surface; the two equations $z=f(x, y)$ and $y=$ constant represent the curve in which the plane, $y=$ constant, cuts the surface. The derivative $\partial z/\partial x$ may therefore be interpreted as the gradient at a point on the curve; the derivative $\partial z/\partial y$ may be interpreted in a similar way.

If $u=x^3\cos y$, then

$$\frac{\partial u}{\partial x}=3x^2\cos y, \quad \frac{\partial^2 u}{\partial x^2}=6x\cos y, \quad \frac{\partial^3 u}{\partial x^3}=6\cos y, \ \ldots,$$

$$\frac{\partial u}{\partial y}=-x^3\sin y, \quad \frac{\partial^2 u}{\partial y^2}=-x^3\cos y, \quad \frac{\partial^3 u}{\partial y^3}=x^3\sin y, \ \ldots$$

The y-derivative of $\frac{\partial u}{\partial x}$ is denoted by

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right), \quad \text{contracted to } \frac{\partial^2 u}{\partial y\,\partial x};$$

the x-derivative of $\frac{\partial u}{\partial y}$ is, in the same way,

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right), \quad \text{contracted to } \frac{\partial^2 u}{\partial x\, \partial y}.$$

Now $$\frac{\partial^2 u}{\partial y\, \partial x}=\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial}{\partial y}(3x^2\cos y)=-3x^2\sin y,$$

$$\frac{\partial^2 u}{\partial x\, \partial y}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)=\frac{\partial}{\partial x}(-x^3\sin y)=-3x^2\sin y.$$

The two derivatives are therefore equal, that is, the result is the same whether we differentiate u first as to x and then as to y or first as to y and then as to x. In other words, the *order* of differentiating is indifferent in this example; it can be proved that the order is indifferent in the case of all the usual functions, and the student may verify the statement for particular cases, *e.g.*

(i) $u=x\sin y+y\sin x$; (ii) $u=e^{ax}\cos by$; (iii) $u=\tan^{-1}(y/x)$.

We now give an important extension of § 23. If $u=f(x, y)$ and if x, y take the increments δx, δy respectively, then the first approximation to δu, the corresponding increment of u, is given by the equation

$$\delta u=\frac{\partial f(x, y)}{\partial x}\delta x+\frac{\partial f(x, y)}{\partial y}\delta y. \quad \text{.........................}(\alpha)$$

We have

$$\delta u=f(x+\delta x, y+\delta y)-f(x, y)$$
$$=\{f(x+\delta x, y+\delta y)-f(x, y+\delta y)\}+\{f(x, y+\delta y)-f(x, y)\}.$$

Consider first the part $f(x, y+\delta y)-f(x, y)$. In passing from the second term to the first, y changes to $y+\delta y$, but x does not change: hence, by § 23, we have

$$f(x, y+\delta y)-f(x, y)=\frac{\partial f(x, y)}{\partial y}\delta y+A(\delta y)^2$$

Next, in passing from the second term of

$$f(x+\delta x, y+\delta y)-f(x, y+\delta y)$$

to the first term, x changes to $x+\delta x$, but $y+\delta y$ does not change; hence

$$f(x+\delta x, y+\delta y)-f(x, y+\delta y)=\frac{\partial f(x, y+\delta y)}{\partial x}\delta x+B(\delta x)^2.$$

The right side of this equation may be put in the form

$$\left(\frac{\partial f(x, y)}{\partial x}+C\,\delta y\right)\delta x+B(\delta x)^2,$$

since $\frac{\partial f(x, y+\delta y)}{\partial x}$ becomes $\frac{\partial f(x, y)}{\partial x}$ when $\delta y=0$. Combining these results, we find

$$\delta u=\frac{\partial f(x, y)}{\partial x}\delta x+\frac{\partial f(x, y)}{\partial y}\delta y+[A(\delta y)^2+C\,\delta y\,\delta x+B(\delta x)^2], \quad ...(\alpha')$$

and therefore, by rejecting the squares and the product of δx and δy, we deduce equation (α).

The increments δx, δy are arbitrary, and are usually denoted by dx, dy. The sum

$$\frac{\partial f(x, y)}{\partial x}dx + \frac{\partial f(x, y)}{\partial y}dy \quad \text{or} \quad \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

is called the first approximation to δu, or *the differential of* u, and is denoted by du, so that

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy. \quad \text{.............................}(\beta)$$

In the same way, if u is a function of three variables x, y, z, the first approximation or differential du is given by

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz, \quad \text{.......................}(\gamma)$$

and so on.

The term $\frac{\partial u}{\partial x}dx$ is called the *partial differential* of u with respect to x; the equations (β), (γ) show that the *total differential du* (or the *complete* differential du) is the *sum* of the partial differentials. In Mechanics the theorem expressed by these equations appears as "the principle of the superposition of small motions."

If x, y are functions of a variable t, we may suppose that the increments δx, δy are due to an increment δt in t; the increment δu in u would therefore be due to the increment δt in t, the function u being now a function of t. Cf. § 26, I. Dividing equation (α') by δt and taking the limit for $\delta t = 0$, we find

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}. \quad \text{...........................}(\delta)$$

Similarly, if u is a function of x, y, z and x, y, z functions of t,

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}. \quad \text{.......................}(\epsilon)$$

In (δ) we may take t to be identical with x; y, and therefore u, would now be functions of x, and we should have

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}. \quad \text{.............................}(\delta')$$

Here du/dx and $\partial u/\partial x$ have quite different meanings, and the student should see that he understands the difference.

If u = constant is the equation of a curve, du/dx is zero and (δ') gives for the gradient

$$\frac{dy}{dx} = -\frac{\partial u}{\partial x}\Big/\frac{\partial u}{\partial y}.$$

For example, if $u = ax^2 + by^2$ and the constant is c, we find for the gradient at (x, y) on the conic $ax^2 + by^2 = c$,

$$\frac{dy}{dx} = -\frac{2ax}{2by} = -\frac{ax}{by}.$$

A few examples for practice in partial differentiation will be found on page 248.

MISCELLANEOUS EXAMPLES.

Chapters I.–IV.

1. At any point P on the curve $y=ax^n$ (n a positive integer) the ordinate MP is drawn, and a point T taken on OX between O and M such that $TM=OM/n$; show that TP is the tangent to the curve at P.

2. Show that the curve $y=ax^3+bx^2+cx+d$ has one, and only one, point of inflexion, and that it lies midway between the turning points, if any. A certain cubic curve passes through the point (0, 20), has a turning point at (2, -8), and a point of inflexion where $x=\frac{3}{4}$; find the equation of the curve, and sketch its graph.

3. Find the gradient of the tangent to the curve

$$y=ax^3+bx^2+cx+d$$

at its point of inflexion, and show that if that tangent is parallel to OX, then y' is a perfect square. Draw a rough sketch of the curve in this case, assuming a to be positive.

4. A handbill is to be printed containing 324 sq. cms. of text, and having margins of 4 cms. each top and bottom, and $2\frac{1}{4}$ cms. each at the sides. Find the dimensions of the bill so that the cost for paper may be as small as possible.

5. Determine the turning points and the points of inflexion on the curve $y=2x^4-12x^2+16x+35$, also the gradient of the curve at each of the latter. Sketch the graph of the curve, and in a separate diagram show clearly how it meets its tangent at the point where $x=-1$.

6. Show that a point of inflexion on a curve is a point where the concavity of the curve changes direction; prove that the curve

$$y=x^4-2x^3-36x^2+7$$

is everywhere concave upwards except between the ordinates $x=-2$ and $x=3$. [See Ex. 54, p. 47.] Sketch the curve.

7. Assuming the formulae for the volume V, and the superficial area S, of a sphere of radius r, show that $dV/dr=S$, and explain why this should be so.

8. The radius of a sphere and the base-radius of a cone of constant altitude h are equal and increasing at the same rate. Find the common radius in terms of h at the instants when their volumes are increasing at the same rate.

9. A horizontal trough with vertical ends is of V-shaped cross-section, the angle between the sides being 60°, and the length of the trough 6 feet. If water enters at the rate of 4 cubic feet per minute, find the rate at which the surface is rising when the depth is 1 foot.

10. Prove that, if u_1, u_2, v_1, v_2 represent functions of x,

$$\frac{d}{dx}\left(\frac{u_1 u_2}{v_1 v_2}\right)=\frac{u_1 u_2}{v_1 v_2}\left\{\left(\frac{1}{u_1}\frac{du_1}{dx}-\frac{1}{v_1}\frac{dv_1}{dx}\right)+\left(\frac{1}{u_2}\frac{du_2}{dx}-\frac{1}{v_2}\frac{dv_2}{dx}\right)\right\}.$$

11. Prove that, if $y=(ax+b)/(cx+d)$, $2y'y'''=3y''^2$; and that, if $a+d=0$, $(y-x)y''=2y'(1+y')$.

12. If $t=a+bx+cx^2$, show that $\ddot{x}/\dot{x}^3$ is constant.

13. If $x=r\cos\theta$ and $y=r\sin\theta$, where x, y, r and θ are functions of t, prove by differentiation that

(i) $\dot{x}\cos\theta+\dot{y}\sin\theta=\dot{r}$, $\quad \dot{y}\cos\theta-\dot{x}\sin\theta=r\dot{\theta}$;

(ii) $\ddot{x}\cos\theta+\ddot{y}\sin\theta=\ddot{r}-r\dot{\theta}^2$, $\quad \ddot{y}\cos\theta-\ddot{x}\sin\theta=\dfrac{1}{r}\dfrac{d}{dt}(r^2\dot{\theta})$.

Show that the first of these equations may be written $x\dot{x}+y\dot{y}=r\dot{r}$, and re-write the others in the same form.

14. With the notation of the preceding question, show that if $r^2\dot{\theta}$ is constant, then $x\ddot{y}=\ddot{x}y$.

15. A ship B is 75 nautical miles due east of a ship A. B sails west at 9 knots and A south at 12 knots. Find their least distance apart and the time of its occurrence.

16. If y is a function of x, and if $z=x^3-3axy+y^3$, show that

$$\frac{dz}{dx}=3(y^2-ax)\frac{dy}{dx}+3(x^2-ay).$$

If x varies in such a way that z is always zero, find dy/dx, and give a graphical interpretation.

17. If $4x^2+9y^2+16x-18y-11=0$, calculate dy/dx, and find the points on the graph of the equation at which the tangent is (i) parallel, (ii) perpendicular to the x-axis.

18. Find the gradient at any point on the curve $(x^2+y^2)^2=4ax^2y$, and state the points on the curve at which the tangent is (i) parallel, (ii) perpendicular to the x-axis.

19. Calculate the gradient for the following curves:

(i) $x(x^2+y^2)=a(x^2-y^2)$; $\quad$ (ii) $(x^2+y^2)=xy(x+y)$.

20. If $ax^2+2hxy+by^2=1$, show that

$$\frac{d^2y}{dx^2}=\frac{h^2-ab}{(hx+by)^3}.$$

21. Differentiate the equation $y^3-3ax^2+x^3=0$ twice, and from the resulting equations show that for this curve

$$\frac{d^2y}{dx^2}+\frac{2a^2x^2}{y^5}=0.$$

Hence show that the curve is everywhere concave towards the x-axis. [See Ex. 54, p. 47.]

22. If $x=t^2/(1+t)$ and $y=t^3/(1+t)$, express D_xy and D_x^2y in terms of t.

Chapters V.–VIII.

23. Show that the substitution $y=x-2$ resolves $\frac{2x^3-6x-7}{(x-2)^4}$ into its component partial fractions, and integrate this function. Evaluate

$$\int_0^1 \frac{x^6\,dx}{(x+1)^5}.$$

24. Sketch the curve $ay=(\alpha-x)(x-\beta)$, a being positive and $\alpha>\beta$, and show that the area cut off by the x-axis is $(\alpha-\beta)^3/6a$.

25. Show that the first quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is divided by the line $\frac{y}{b}=\frac{\sqrt{3}x}{a}$ into two parts, one of which is a third of the quadrant.

26. The friction in a journal is given by the equation

$$F=k\int_0^{\frac{\pi}{2}}\sqrt{(\cos\theta)}\,d\theta.$$

Evaluate the integral by Simpson's rule for 11 ordinates.

27. A cylindrical hole is driven through a solid sphere of radius R, the axis of the cylinder coinciding with a diameter of the sphere and its radius being r. Calculate the volume of the part of the sphere left.

28. A section consists of a semi-circular area ABC, radius 1·5 inches, surmounting a rectangle $DEFG$, the whole figure being symmetrical about the line through the centre of the circle parallel to DE or FG. If $DE=3$ inches and $EF=1$ inch, find the position of the centroid and the moment of inertia of the section about the line of symmetry BH, where H lies on the lower edge of the rectangle.

29. A plane lamina in the form of a parabola is lowered, with its axis vertical and its vertex downwards, into water; find the depth of the centre of pressure when the vertex is at a depth h, and show that, if the lamina descends with uniform velocity 7 inches per second, the C.P. descends with uniform velocity 4 inches per second.

30. A triangular lamina is immersed in a liquid with one side in the free surface, its plane making an angle θ with the vertical. Prove that, if the triangle be divided into two parts by a horizontal line through its centre of pressure, the thrusts on the two parts are equal. [Use the result of Ex. 2, p. 94.]

31. A piece of sheet metal is cut in the form of the segment of a parabola cut off by a double ordinate at a distance h from its vertex. If $2k$ is the length of the straight edge, M the mass of the piece, prove that its moment of inertia (i) about the tangent at the vertex is $\frac{3}{7}Mh^2$, (ii) about the axis of the parabola is $\frac{1}{5}Mk^2$.

32. Prove that the moment of inertia of the solid anchor-ring described in Ex. 8, p. 87, about its axis of revolution, is $M(c^2+\frac{3}{4}a^2)$, where M is the mass of the ring.

33. A plane lamina is symmetrical about OX, one of two rectangular axes OX, OY in its plane. Any line is drawn through O, in that plane, making an angle θ with OX: prove that the moment of inertia of the lamina about this line is $I_{OX}\cos^2\theta+I_{OY}\sin^2\theta$.

Deduce the M.I. of a rectangle, sides $2a$ and $2b$, about a diagonal.

34. Find, in cubic feet, the volume occupied by one pound of dry air when at a pressure of 90 lbs. per square inch, and at a temperature of 60° F.

Find also the work done by this volume of air when it expands adiabatically to a volume such that the pressure is 15 lbs. per square inch.

35. A quantity of air compressed to ten atmospheres is made to expand isothermally down to one atmosphere. Calculate in atmospheres the mean pressure during the expansion.

36. The graph of a polynomial of odd degree, $2n+1$, in x crosses OX at the points where $x=0, 1, 2, 3 \dots 2n$; show that the area between the graph and OX from $x=0$ to $x=2n$ is zero. [First shift the origin to the point $(n, 0)$.]

37. If s denotes the length of an arc of the curve $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$, measured from where it crosses OY to a point whose abscissa is x, prove that $s^3=kx^2$, where k is a constant.

Chapters IX., X.

38. Show that the differential equation of simple harmonic motion, $\frac{d^2x}{dt^2}+n^2x=0$, may be written in the form $\left(\frac{dx}{dt}\right)^2+n^2x^2=\text{const.}$, and state this equation in words. Calculate in C.G.S. units the value of the constant in the case of the bob of a simple pendulum whose period of oscillation is 2 seconds, and which vibrates through a total arc of 3 cms.

39. If $x=a(1-\cos t)$ and $y=a(nt+\sin t)$, express $\frac{d^2y}{dx^2}$ in terms of t.

40. A point travels round the circle $x^2+y^2=a^2$ with uniform angular velocity ω; find its angular velocity about the point $(b, 0)$, $b>a$, at the instants when it crosses the axes.

41. AOB is a quadrant of a circle, centre O; P is taken on the arc AB and Q on the radius OA, so that $\angle AOP=\angle OPQ=\theta$. Show that the area of the triangle QAP is $\frac{1}{4}a^2(2\sin\theta-\tan\theta)$, and find the position of P when the area is a maximum.

42. Prove that if any segment of the sine-curve $y=a\sin\frac{x}{b}$ cut off by the x-axis revolves once about OX, the volume generated is one half that of the circumscribed cylinder.

43. Prove (i) by interpreting each integral as a second moment, (ii) by evaluating each integral by a trigonometric substitution, that

$$\int_0^a x^2\sqrt{(a^2-x^2)}\,dx=\tfrac{1}{4}\int_0^a (a^2-x^2)^{\frac{3}{2}}\,dx.$$

44. Integrate $$\frac{5\cos x-2\sin x}{2\cos x+5\sin x}.$$

Determine the constants λ, μ in the identity

$$\cos x+17\sin x\equiv\lambda(2\cos x+5\sin x)+\mu(5\cos x-2\sin x),$$

and hence write down the integral of $\frac{\cos x+17\sin x}{2\cos x+5\sin x}$.

Integrate in the same way

$$\frac{5\cos x+4\sin x}{3\cos x-2\sin x},\quad\text{and}\quad\frac{A\cos x+B\sin x}{P\cos x+Q\sin x}.$$

45. Show that the graph of $y^2=(1+x)/(1-x)$ lies entirely between two fixed ordinates, is symmetrical about OX, has no turning points, and has points of inflexion where $x=-\frac{1}{2}$. Determine its gradient where $x=-1$, 0 and $+1$, and sketch the graph.

Find the whole area enclosed by the curve and the asymptote, using the substitution $x=\cos\theta$ for the integration.

46. Find the area between the curve whose equation is $y^2(2a-x)=x^3$ and its asymptote.

Show that the ordinate $x=a$ divides the area into parts in the ratio of $3\pi-8$ to $3\pi+8$.

47. Draw sketches of the graphs of $y=\sin^9 x$, $y=\sin^{10} x$ from $x=-\frac{\pi}{2}$ to $x=\pi$, and write down the values of

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^9 x\,dx,\quad \int_0^{\pi}\sin^9 x\,dx,\quad \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^{10} x\,dx,\quad\text{and}\quad \int_0^{\pi}\sin^{10} x\,dx.$$

48. A particle of mass m, moving with constant acceleration in a straight line, has initial velocity v_0 and final velocity v_1. Prove that its mean kinetic energy, with respect to

(i) the time, is $\frac{1}{6}m(v_0^2+v_0v_1+v_1^2)$,

(ii) the displacement, is $\frac{1}{4}m(v_0^2+v_1^2)$.

49. Prove that the mean distance of points on the perimeter of the ellipse $x^2/a^2+y^2/b^2=1$ from either focus, x being the variable, is equal to the semi-major axis; also that, if the eccentric angle be taken as variable, the mean square of their distances from the centre is $(a^2+b^2)/2$.

50. The density of a sphere at distance r from the centre is

$$\rho_0\left(1-\frac{kr}{a}\right),$$

where a is the radius of the sphere and k is a constant. Find the mean density of the sphere.

51. Determine the period of the function $12\sin 18t+5\cos 30t$, drawing sketches of its terms in one diagram, and calculate its Root Mean Square over a full period.

52. If I_n denote $\int_0^{\frac{\pi}{2}}\sin^n\theta\,d\theta$, where n is a positive integer, show by integration by parts (first writing the integrand $\sin\theta.\sin^{n-1}\theta$) that

$$I_n=\frac{n-1}{n}I_{n-2}.$$

In this formula substitute $I_{n-2}=\frac{n-3}{n-2}I_{n-4}$, then substitute similarly for I_{n-4}, and so on, and thus determine the value of I_n as given in equation (A), p. 127, distinguishing between the cases of n odd and n even.

53. Integrate $x^3/(x^2+1)^3$, (i) by the substitution $u=x^2+1$, (ii) by the substitution $x=\tan\theta$. Explain why you get different results.

Chapters XII., XIII.

54. Show from the cosecant and secant graphs that the principal values of $\text{cosec}^{-1}x$, $\sec^{-1}x$ have the ranges $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$ and 0 to π respectively. Find their derivatives, and show that

$$\text{cosec}^{-1}x+\sec^{-1}x=\frac{\pi}{2}.$$

Differentiate $\sec^{-1}(\sqrt{\sec x})$.

55. Integrate $\int \frac{dx}{3\cos x + 4\sin x}$, first writing the denominator in the form $R\sin(x+\alpha)$. Obtain the same integral by the substitution $u=\tan\frac{1}{2}x$. [See Ex. 37, p. 160.]

Use the first method to integrate $\frac{1}{A\cos x + B\sin x}$.

56. Evaluate $\int \frac{dx}{13+3\cos x+4\sin x}$ by the substitution $u=\tan\frac{1}{2}x$, and $\int \frac{2dx}{\cos^2 x - 2\sin x\cos x + 5\sin^2 x}$ by the substitution $u=\tan x$.

57. Calculate, with the help of tables, correct to three decimals in each case, $\int_0^{\frac{\pi}{2}} \frac{d\theta}{1+e\cos\theta}$, (i) when $e=\frac{24}{25}$, (ii) when $e=\frac{25}{24}$.

58. Determine the integral of $\sqrt{(x^2+1)}/x^4$ by substituting $x=\tan\theta$ and then $u=\sin\theta$. What *single* algebraic substitution would effect the integration?

Evaluate $\int \frac{dx}{2x\sqrt{(x-1)}}$ and $\int \sqrt{\left(\frac{2-x}{x}\right)}\,dx$.

59. Sketch the curve whose equation is $y^2(a+x)=c^2(a-x)$, and find (i) the whole area between the curve and its asymptote, (ii) the area bounded by the curve and OY.

60. Prove that the following expressions are positive for all positive values of x:

(i) $(x-1)e^x+1$; (ii) $(x-2)e^x+x+2$; (iii) $(x-3)e^x+\frac{1}{2}x^2+2x+3$.

61. When a thin plate at uniform temperature v is surrounded by a gas kept at the slightly lower temperature v_0, the plate cools at a rate proportional to $(v-v_0)$; if initially the temperature of the plate is V, prove that after time t it has fallen to $v_0+(V-v_0)e^{-kt}$, where k is the constant of proportionality.

Calculate the missing readings in the following table:

$v_0=50°$ C.	t mins.	0	5	10	
	$v°$ C.	65	60		53

62. Find, in terms of θ, the gradient at any point of the curve given by the equations $x=a\log\cot\frac{1}{2}\theta - a\cos\theta$, $y=a\sin\theta$, and prove that the length of the tangent intercepted between OX and the point of contact is constant.

63. In Fig. 32, p. 175, show that

$$PK=x\sqrt{(1+y'^2)}, \quad \text{and} \quad LK=\left(x-\frac{y}{y'}\right)\sqrt{(1+y'^2)}.$$

Prove that the length of any tangent to the curve $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ intercepted between the coordinate axes is always a.

64. A curve is plotted whose ordinates are the velocities, for the corresponding values of x, of a point moving along OX. If MP is any ordinate of the curve, PG the normal to the curve cutting OX in G, prove that MG is the acceleration of the point as it passes through M.

65. The coordinates of a point on a curve are given as functions of a variable t; prove that the radius of curvature is

$$(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}} \div (\dot{x}\ddot{y}-\ddot{x}\dot{y}).$$

Calculate the curvature at any point of

$$x=a \sin t\,(1+\cos t), \quad y=a \cos t\,(1-\cos t).$$

66. Calculate the radius of curvature, at the origin, of the curve

$$4x^3-3xy^2+17y^2+7xy-5x+12y=0.$$

Prove that if the centre of curvature of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ at one end of the minor axis lies at the other end, then the eccentricity of the ellipse is $\dfrac{1}{\sqrt{2}}$.

67. Sketch the curve $x^2y+4a^2(2a+y)=0$, and calculate its radius of curvature where it crosses OY. Sketch also the circle of curvature at that point, and write down its equation.

68. The **cycloid** (see Ex. 59, p. 112) is the periodic curve traced by a point on the circumference of a circle of radius a which rolls without slipping along OX, the point being initially at the origin. Show that after the circle has rolled through an angle θ the coordinates of the point are $x=a(\theta-\sin\theta)$, $y=a(1-\cos\theta)$, and prove the following properties of the curve:

(i) The *area* under an arch of the cycloid is three times that of the generating circle.
(ii) The *length* of an arch is eight times the radius of the circle.
(iii) The *volume of revolution* of an arch about OX is $5\pi^2a^3$.
(iv) The *superficial area* of this solid of revolution is $\frac{64}{3}\pi a^2$.
(v) The *radius of curvature* is $-4a\sin\frac{1}{2}\theta$.
(vi) The *centroids* of the area under an arch and of the arch itself are at heights $\frac{5}{6}a$ and $\frac{4}{3}a$, respectively, above OX. (Apply Pappus' Theorems.)

69. A horizontal line $y=k$ cuts the cycloid (see preceding example) in two points P and P' between $x=0$ and $x=2\pi a$; prove that

$$\text{arc } OP \times \text{arc } OP' = 8ak.$$

(Note that $\theta_{P'}=2\pi-\theta_P$.)

70. The surface density at any point (x, y) of a thin plate cut in the form of a quadrant of the ellipse $x^2/a^2+y^2/b^2=1$ is proportional to xy. Calculate, by double integration, the mean value of the density of the plate. Find also the coordinates of the centroid of the plate.

71. (i) If $\rho = W/(W-w)$, prove that, approximately,

$$\frac{\delta\rho}{\rho} = \frac{-w}{W-w}\,\frac{\delta W}{W} + \frac{\delta w}{W-w}.$$

(ii) If $S = \frac{1}{2}ab\sin C$, prove that, approximately,

$$\frac{\delta S}{S} = \frac{\delta a}{a} + \frac{\delta b}{b} + \cot C\,\delta C.$$

72. If $u = \log\sqrt{(x^2+y^2)}$ or if $u = \tan^{-1}(y/x)$, prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

73. If $u = e^{-a^2n^2t}(A\cos nx + B\sin nx)$, prove that

$$\frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2}.$$

74. If $u = A\cos m(x+at) + B\sin n(x-at)$, prove that

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}.$$

75. If $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=k$, prove that

$$\frac{\partial z}{\partial x} = -\frac{ax+hy+gz}{gx+fy+cz}, \quad \frac{\partial z}{\partial y} = -\frac{hx+by+fz}{gx+fy+cz}.$$

MISCELLANEOUS INTEGRALS.

1. $\int \frac{1}{x}\sqrt{(x^2-a^2)}\,dx.$ **2.** $\int \frac{2\,dx}{e^x+e^{-x}}.$

3. $\int_0^a \frac{\sqrt{(a^2-x^2)}}{b^2-x^2}\,dx,\ b>a>0.$ [Put $x=a\sin\theta$.]

4. $\int \frac{1-\sin x}{1+\cos x}\,dx.$ **5.** $\int \frac{x+2}{x\sqrt{(x-3)}}\,dx.$

6. $\int_2^6 \frac{dx}{\sqrt{(8x-12-x^2)}}$ [See Ex. 52, p. 112.] **7.** $\int x\log(1+x^2)\,dx.$

8. $\int_0^\infty \frac{x^5\,dx}{(x^2+a^2)^5}.$ **9.** $\int \frac{\sqrt{(1-x^2)}}{1+x^2}\,dx.$ **10.** $\int \sqrt{\left(\frac{x-a}{b-x}\right)}\,dx$

11. $\int_0^a (a^{\frac{2}{3}}-x^{\frac{2}{3}})^{\frac{3}{2}}\,dx.$ [Substitute $x=a\sin^3\theta$.] **12.** $\int_1^5 \frac{(x+1)\,dx}{(x^2-6x+13)^2}.$

13. $\int 4\sin x\sin 2x\sin 3x\,dx.$ **14.** $\int \frac{x\,dx}{\sqrt{(1-x-x^2)}}$

15. $\int_0^\infty \frac{\sin 4t}{e^{3t}}\,dt.$ **16.** $\int \frac{d\theta}{\sin\theta(\sin\theta+\cos\theta)}.$ [Algebraic in $\tan\theta$.]

17. $\int \frac{dx}{\sqrt{\{(x-1)(x-3)\}}}.$ **18.** $\int \frac{x^{\frac{3}{2}}}{\sqrt{(a-x)}}\,dx.$

19. $\int \frac{d\theta}{a+b\tan\theta}.$ [Multiply above and below by $\cos\theta$; then use the device of Ex. 44, Misc. Exs., p. 244.]

20. $\int_4^6 \frac{(x-1)(3-x)}{\sqrt{\{(x-4)(6-x)\}}}\,dx.$ **21.** $\int \frac{dx}{25-24\sin^2 x}.$ [Express in terms of $u\equiv 2x$.]

22. $\int \frac{x\,dx}{(1+x^2)\sqrt{(1-x^2)}}.$ **23.** $\int \frac{x^2\,dx}{\sqrt{(1-x^6)}}.$

24. $\int_0^\infty \frac{dx}{(1+x)^{\frac{3}{2}}+(1+x)^{\frac{1}{2}}}.$ **25.** $\int \tan^{-1}x\,dx.$

26. $\int xe^x \sin x\, dx$. [Integrate by parts, taking $e^x \sin x$ as the integrable part. See Ex. 1, § 62.]

27. $\int_2^{2\frac{3}{4}} \sqrt{(4x^2 - 7x - 2)}\, dx$.

28. $\int_0^{\frac{a}{\sqrt{2}}} \frac{x^2 + ax + a^2}{(a^2 - x^2)^{\frac{3}{2}}}\, dx$.

29. $\int \frac{dx}{(x-a)\sqrt{(x^2 - 2ax)}}$.

30. $\int \frac{\sqrt{(x-1)}}{(x-2)(x-5)}\, dx$.

31. $\int \frac{3 + 4\cos\theta}{3 + 5\cos\theta}\, d\theta$. $\left[\text{Express integrand in form } a + \frac{b}{3 + 5\cos\theta}.\right]$

32. $\int_0^{\frac{\pi}{2}} \frac{\cos^2\theta\, d\theta}{(1 + 3\sin^2\theta)^2}$. [Change variable to ϕ, where $\tan\phi = 2\tan\theta$.]

33. $\int_3^8 x\sqrt{(16 + 6x - x^2)}\, dx$.

34. $\int \frac{\sin^{-1}x\, dx}{(1 - x^2)^{\frac{3}{2}}}$.

35. $\int_0^{\frac{\pi}{2}} \frac{dx}{(9\cos^2 x + 16\sin^2 x)^2}$. [Cf. Ex. 21, p. 173.]

36. $\int \frac{dx}{x - \sqrt{(x^2 - a^2)}}$. [First rationalise the denominator.]

37. $\int \frac{dx}{\cos^3 x}$.

38. $\int_1^\infty \frac{dx}{(x+2)\sqrt{(x-1)}}$.

39. $\int_0^{\frac{\pi}{3}} \tan\theta \sec^3\theta\, d\theta$.

40. $\int \frac{\tan x\, dx}{a + b\tan^2 x}$. [Cf. 19.]

41. $\int \frac{P\, d\theta}{1 + e^\theta}$, where P has the values (i) e^θ, (ii) 1, (iii) $e^{2\theta}$, (iv) $e^{-\theta}$.

42. $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \operatorname{cosec}^4\theta\, d\theta$.

43. $\int \frac{x\, dx}{a^4 - x^4}$.

44. $\int \frac{dx}{7 + 5\cos x + 3\sin x}$. [See Ex. 56, Misc. Exs., p. 246.]

45. $\int_0^a (x^2 + a^2)^{-3}\, dx$.

46. $\int \frac{dx}{3x + 5x^7}$.

47. $\int_0^\infty \frac{dx}{1 + 2x\cos\theta + x^2}$. Discuss the case $\theta = 0$.

48. $\int \{x(1 + x^{\frac{1}{2}})\}^{\frac{1}{2}}\, dx$.

49. $\int \frac{(2x^2 - 3)\, dx}{(x^2 - 4)(x^2 + 9)}$.

50. $\int \frac{dx}{(x^2 + m^2)\sqrt{(x^2 + n^2)}}$, (i) when $m^2 < n^2$, (ii) when $m^2 > n^2$.

[Substitute $x = n\tan\theta$, and then $\sin\theta = u$ (or, $x = nu/\sqrt{(1 - u^2)}$, in one operation).]

ANSWERS.

Exercises. I. Page 17.

1. 2, 0, 8, 14.

2. 1, $1\frac{7}{8}$, $\frac{5}{8}$.

3. $3a^2x^2+2a(3b+1)x+3b^2+2b-1$, $3x^4+2x^2-1$, $3x^6+2x^3-1$.

4. 0, 0·5, 0·7373, 1.

5. 4, 3, -4, 4·686, 4·948.

7. 3.

8. a.

9. $3ax_1^2+b+3ax_1\delta x_1+a(\delta x_1)^2$.

10. $3ax_1^2+2bx_1+c+(3ax_1+b)\delta x_1+a(\delta x_1)^2$.

11. $-\dfrac{2x_1+\delta x_1}{x_1^2(x_1+\delta x_1)^2}$.

12. $\dfrac{1}{\delta x_1}\log\left(1+\dfrac{\delta x_1}{x_1}\right)$.

13. $10^{x_1}\left\{\dfrac{10^{\delta x_1}-1}{\delta x_1}\right\}$.

14. $10^{-x_1}\left\{\dfrac{10^{-\delta x_1}-1}{\delta x_1}\right\}$.

15. (i) 0·851, 0·857, 0·860; (ii) $-0·691$, $-0·685$, $-0·676$.

16. (i) $-0·521$, $-0·516$, $-0·504$; (ii) $-0·721$, $-0·728$, $-0·734$.

17. (i) 1·374, 1·361, 1·347; (ii) 2·088, 2·129, 2·166.

20. $\cos x$, $\sec x$, 10^x+10^{-x}, $(10^x-10^{-x})/x$ are even; the other functions are odd.

Exercises. II. Page 31.

1. 5. **2.** 5. **3.** $8x$. **4.** $8x$.

5. $14x-12$. **6.** $11-4x$. **7.** $9x^2-8x-1$. **8.** $-4x+12x^2-20x^4$.

9. $2x+3$. **10.** $12x+5$. **11.** $3x^2+12x+11$. **12.** $2acx+af+bc$.

13. $1-\dfrac{1}{x^2}$. **14.** $2x-\dfrac{2}{x^3}$. **15.** $4+\dfrac{5}{x^2}$. **16.** $a-\dfrac{c}{x^2}$.

17. $6x+2-\dfrac{1}{x^2}$. **18.** $6(2x+1)^2$. **19.** $10(2x+1)^4$. **20.** $-3(1-x)^2$.

21. $-4(3-x)^3$. **22.** $-\dfrac{1}{(x+1)^2}$. **23.** $\dfrac{1}{(1-x)^2}$. **24.** $-\dfrac{4}{(2x+3)^3}$.

25. $\dfrac{4}{(3-2x)^3}$. **26.** $\dfrac{1}{\sqrt{(2x+3)}}$. **27.** $-\dfrac{1}{\sqrt{(2x+3)^3}}$. **28.** $-\dfrac{1}{\sqrt[3]{(2-3x)^2}}$.

29. $2x-5$. **30.** $1-\dfrac{18}{(x-1)^3}$. **31.** $10-6t$. **32.** $-\dfrac{720}{v^2}$.

33. $-\dfrac{1080}{v^{\frac{5}{2}}}$. **34.** $-\dfrac{4}{(v+3)^2}$. **35.** $b+2ct$. **36.** $\dfrac{3}{2\sqrt{(3u-4)}}$.

37. $-\dfrac{3}{2\sqrt{(3u-4)^3}}$. **38.** $3(2u-5)$. **39.** $a-\dfrac{c}{t^2}$. **40.** $-\dfrac{1\cdot 4c}{v^{2\cdot 4}}$.

41. $x^3+\dfrac{x^2}{2}-x$. **42.** $\dfrac{x^2}{2}+x+\dfrac{1}{x}$. **43.** $\dfrac{2x^3}{3}-\dfrac{1}{2x}$. **44.** $2\sqrt{x}$.

45. $4\sqrt{x}+\dfrac{6}{\sqrt{x}}-\dfrac{3}{x^2}$. **46.** $\dfrac{(x+1)^3}{3}$. **47.** $\dfrac{(3x+2)^3}{9}$.

48. $-\dfrac{1}{3(3x+2)}$. **49.** $\dfrac{2\sqrt{(3x+2)}}{3}$. **50.** $-\dfrac{1}{2(2x-3)}$.

Exercises. III. Page 39.

1. Max. $28\frac{3}{4}$ at $x=2\frac{1}{2}$.

2. Max. $21\frac{1}{8}$ at $x=1\frac{3}{4}$.

3. Max. 4 at $x=-1$.
Min. 0 at $x=1$.
Point of inflexion (0, 2).

4. Max. $\frac{3}{4}$ at $x=\frac{1}{2}$.
Min. -6 at $x=2$.
Point of inflexion $(\frac{5}{4}, -2\frac{5}{8})$.

5. Max. 17 at $x=-1$. Min. -10 at $x=2$. Point of inflexion $(\frac{1}{2}, 3\frac{1}{2})$.

6. Min. -30 at $x=-1$. Max. 2 at $x=1$. Min. -3 at $x=2$.
Points of inflexion $\left(\dfrac{2-\sqrt{7}}{3}, -\dfrac{230+80\sqrt{7}}{27}\right)$, $\left(\dfrac{2+\sqrt{7}}{3}, -\dfrac{230-80\sqrt{7}}{27}\right)$

7. Min. -11 at $x=1$. Points of inflexion $(0, -10)$, $(\frac{2}{3}, -10\frac{16}{27})$.

8. Max. 11 at $x=1$. Min. -17 at $x=3$. Points of inflexion
$(0, 10)$, $\left(\dfrac{3-\sqrt{3}}{2}, \dfrac{17+39\sqrt{3}}{8}\right)$, $\left(\dfrac{3+\sqrt{3}}{2}, \dfrac{17-39\sqrt{3}}{8}\right)$.

9. Max. $\frac{27}{16}$ at $x=\frac{1}{2}$. Points of inflexion (0, 1), $(-1, 0)$.

10. Max. $\frac{108}{3125}$ at $x=\frac{3}{5}$. Min. 0 at $x=1$. Points of inflexion
$(0, 0)$, $\left(\dfrac{6-\sqrt{6}}{10}, \dfrac{1656+84\sqrt{3}}{100000}\right)$, $\left(\dfrac{6+\sqrt{6}}{10}, \dfrac{1656-84\sqrt{3}}{100000}\right)$.

11. Max. $\dfrac{2\sqrt{3}}{9}$ at $x=\dfrac{6-\sqrt{3}}{3}$.
Min. $-\dfrac{2\sqrt{3}}{9}$ at $x=\dfrac{6+\sqrt{3}}{3}$.
Point of inflexion (2, 0).

12. Min. $-6\sqrt{3}$ at $x=-\sqrt{3}$.
Max. $6\sqrt{3}$ at $x=\sqrt{3}$.
Point of inflexion (0, 0).

13. Max. $20\frac{1}{4}$ at $x=-\dfrac{3\sqrt{2}}{2}$. Min. 0 at $x=0$. Max. $20\frac{1}{4}$ at $x=\dfrac{3\sqrt{2}}{2}$.
Points of inflexion $\left(-\dfrac{\sqrt{6}}{2}, \dfrac{45}{4}\right)$, $\left(\dfrac{\sqrt{6}}{2}, \dfrac{45}{4}\right)$.

14. Min. $-\dfrac{1458\sqrt{15}}{125}$ at $x=-\dfrac{3\sqrt{15}}{5}$. Max. $\dfrac{1458\sqrt{15}}{125}$ at $x=\dfrac{3\sqrt{15}}{5}$.
Points of inflexion
$(0, 0)$, $\left(-\dfrac{3\sqrt{30}}{10}, -\dfrac{5103\sqrt{30}}{1000}\right)$, $\left(\dfrac{3\sqrt{30}}{10}, \dfrac{5103\sqrt{30}}{1000}\right)$.

15. Max. -9 at $x=-3$. Min. 15 at $x=3$.
16. Max. -1 at $x=-1$. Min. 7 at $x=3$. **17.** Min. 9 at $x=2$.
18. Max. -20 at $x=1$. Min. -23 at $x=2$.

Points of inflexion where $x=-\sqrt[4]{\frac{24}{7}}$ and $\sqrt[4]{\frac{24}{7}}$.

24. $\frac{\sqrt{3}}{3}d$; $\frac{\sqrt{6}}{3}d$. **25.** $\frac{1}{6}(a+b-\sqrt{a^2-ab+b^2})$. **26.** $3^3\times 2^2\times\left(\frac{k}{5}\right)^5$.

27. $5^5\times 3^3\times\left(\frac{k}{8}\right)^8$; $\left(\frac{x}{m}\right)^m\left(\frac{y}{n}\right)^n<\left(\frac{x+y}{m+n}\right)^{m+n}$; $a^m b^n<\left(\frac{ma+nb}{m+n}\right)^{m+n}$,

unless $\frac{x}{m}=\frac{y}{n}$ or $a=b$.

Exercises. IV. Page 45.

1. $2dx$. **2.** adx. **3.** $6xdx$. **4.** $2axdx$.
5. $2xdx$. **6.** $2xdx$. **7.** $-2xdx$. **8.** $(6x-4)\,dx$.
9. $(2ax+b)\,dx$. **10.** $\frac{1}{2\sqrt{x}}dx$. **11.** $-\frac{1}{2\sqrt{x^3}}dx$.
12. $\left(2ax-\frac{2b}{x^3}\right)dx$. **13.** $n\left(ax^{n-1}-\frac{b}{x^{n+1}}\right)dx$. **14.** $\frac{1}{2}x^2$.
15. $\frac{1}{2}(x+1)^2$. **16.** $\frac{1}{9}(3x+1)^3$. **17.** $-\frac{1}{x}$.
18. $2\sqrt{x}$. **19.** $-\frac{2}{\sqrt{v}}$. **20.** $-\frac{1}{0{\cdot}4v^{0{\cdot}4}}$.
21. $-\frac{1}{v+1}$. **22.** $2\sqrt{v+1}$. **23.** $\left(\frac{1}{2}u^2+2u-\frac{1}{u}\right)$.
24. $\frac{z^2+4}{2z}$. **25.** $\frac{2}{3}\sqrt{y^3}+2\sqrt{y}$. **26.** $3x^2-x-3+(6x-1)dx$.
27. 1·5516. **28.** 2·1378. **29.** 2·1889.
37. 6; 0. **38.** $120x^2-90x+24$; $240x-90$.
39. $\frac{2}{x^3}$; $-\frac{6}{x^4}$. **40.** $\frac{6}{x^4}$; $-\frac{24}{x^5}$.
41. $\frac{2}{(x+1)^3}$; $-\frac{6}{(x+1)^4}$. **42.** $\frac{6}{(x+1)^4}$; $-\frac{24}{(x+1)^5}$.
43. $\frac{2c}{x^3}+\frac{6d}{x^4}$; $-\frac{6c}{x^4}-\frac{24d}{x^5}$. **44.** $-\frac{1}{4\sqrt{x^3}}$; $+\frac{3}{8\sqrt{x^5}}$.
45. $\frac{3}{4\sqrt{x^5}}$; $-\frac{15}{8\sqrt{x^7}}$. **46.** $\frac{n(n+1)}{x^{n+2}}$; $-\frac{n(n+1)(n+2)}{x^{n+3}}$.
47. $2a^2-12ax+12x^2$; $-12a+24x$.
48. $6x(a^3-6a^2x+10ax^2-5x^3)$; $6(a^3-12a^2x+30ax^2-20x^3)$.
49. (0, 3). **50.** (0, 5). **51.** $\left(-\frac{4}{15}, 2\frac{668}{675}\right)$.
52. $(-2, 0)$. **53.** $\left(\frac{3-\sqrt{3}}{6}a, \frac{a^4}{36}\right)$, $\left(\frac{3+\sqrt{3}}{6}a, \frac{a^4}{36}\right)$.

Exercises. V. Page 54.

1. (i) 400; $300-32t$. (ii) 0; -32. $(3750,\ 1406\frac{1}{4})$; $\frac{3}{4}-\frac{2}{25}t$; $\frac{3}{4}$.

2. (i) U; $V-gt$. (ii) 0; $-g$. $\left(\frac{UV}{g},\ \frac{V^2}{2g}\right)$; $\frac{V}{U}-\frac{gt}{U}$; $\frac{V}{U}$.

3. $a\frac{d^2\theta}{dt^2}$.

4. $-dN/dt$ is the time-rate of *decrease* of the number of lines that pass through the circuit; *or* the time-rate at which lines are *withdrawn from* the circuit.

5. $E=RC+L\frac{dC}{dt}$.

6. $\frac{x}{\sqrt{(x^2-1)}}$.

7. $-\frac{x}{\sqrt{(1-x^2)}}$.

8. $\frac{x}{\sqrt{(x^2+a^2)}}$.

9. $-\frac{x}{\sqrt{(a^2-x^2)}}$.

10. $\frac{3x}{\sqrt{(3x^2+5)}}$.

11. $-\frac{3x}{\sqrt{(5-3x^2)}}$.

12. $\frac{ax}{\sqrt{(ax^2+b)}}$.

13. $-\frac{ax}{\sqrt{(b-ax^2)}}$.

14. $\frac{x+1}{\sqrt{(x^2+2x-3)}}$.

15. $\frac{3x-2}{\sqrt{(3x^2-4x+5)}}$.

16. $\frac{2-3x}{\sqrt{(5+4x-3x^2)}}$.

17. $\frac{2ax+b}{2\sqrt{(ax^2+bx+c)}}$.

18. $-\frac{x}{\sqrt{(x^2+1)^3}}$.

19. $\frac{x}{\sqrt{(1-x^2)^3}}$.

20. $\frac{5(1-x)}{\sqrt{(5x^2-10x+6)^3}}$.

21. $-\frac{2ax+b}{2\sqrt{(ax^2+bx+c)^3}}$.

22. $\frac{2x}{3\sqrt[3]{(x^2+1)^2}}$.

23. $\frac{ax^2}{\sqrt[3]{(ax^3+b)^2}}$.

24. $-\frac{ax^2}{\sqrt[3]{(ax^3+b)^4}}$.

25. $-\frac{3ax^2+b}{3\sqrt[3]{(ax^3+bx+c)^4}}$.

26. $3aAx^2+2(aB+bA)x+aC+bB$.

27. $4x^3-20x$.

28. $3x^2+12x+11$.

29. $2(2x-1)(3x+4)(12x+5)$.

30. $(ax+b)(cx+d)^2(5acx+2ad+3bc)$.

31. $(Ax^2+Bx+C)(5aAx^2+3aBx+4Abx+aC+2bB)$.

32. $\frac{1-2x^2}{\sqrt{(1-x^2)}}$.

33. $\frac{4x^2+3x+8}{\sqrt{(x^2+4)}}$.

34. $\frac{a^2-ax-2x^2}{\sqrt{(a^2-x^2)}}$.

35. $\frac{2aAx^2+Abx+aB}{\sqrt{(Ax^2+B)}}$.

36. $\frac{(3x+4)(18x^2+53x-12)}{2\sqrt{(x^2+3x-2)}}$.

37. $\frac{2x+1}{2\sqrt{(x+3)(x-2)}}$.

38. $\frac{-2}{(5x+6)^2}$.

39. $\frac{ad-bc}{(cx+d)^2}$.

40. $\frac{1-2x-x^2}{(x^2+1)^2}$.

41. $\frac{2x}{(x^2+1)^2}$.

42. $\frac{1-6x-8x^2}{(2+5x+x^2)^2}$.

43. $\frac{30x^2+8x-8}{x^2(2-x)^2}$.

44. $\frac{x+2}{2\sqrt{(x+1)^3}}$.

45. $\frac{1}{(x+1)\sqrt{(x^2-1)}}$.

46. $\frac{1-x^2}{(x^2-x+1)\sqrt{(x^4+x^2+1)}}$.

47. $\frac{bx+2c}{2\sqrt{(ax^2+bx+c)^3}}$.

48. $vw\frac{du}{dt}+wu\frac{dv}{dt}+uv\frac{dw}{dt}$ cubic feet per sec. Formula (B').

49. (i) 4π feet per minute; (ii) 40π square feet per min.

50. $2\pi xv$ square feet per min.

51. (i) $8\pi xv$ square feet per min.; (ii) $4\pi x^2v$ cubic feet per min.

52. $\frac{12}{25\pi}$ feet per min.; $\frac{48}{\pi x^2}$ feet per min.

53. $\frac{1}{5}$ foot per min.

54. Max. $\frac{25}{2}$; Min. $-\frac{25}{2}$.

55. Max. $\frac{3\sqrt{3}}{4}a^2$; Min. $-\frac{3\sqrt{3}}{4}a^2$.

56. Max. $\frac{4^4\times 3^3}{7^7}$; Min. 0.

57. Min. $-\frac{1}{2}$; Max. $\frac{1}{2}$.

58. Neither maximum nor minimum; but the function has 1 for upper limit and -1 for lower limit.

59. Min. $\frac{1}{3}$; Max. 3.

60. Min. $-\frac{1}{2(a-b)}$; Max. $\frac{1}{2(a+b)}$.

61. Max. $\frac{4\sqrt{2}}{3\sqrt{3}}a^{\frac{3}{2}}$.

62. $\frac{1}{2}d$; $\frac{\sqrt{3}}{2}d$.

64. $a/\sqrt{2}$.

65. $AP:PB=a:b$.

66. $\frac{1}{2}u^{\frac{1}{2}}$; $\frac{1}{3}(x^2+2x+2)^{\frac{3}{2}}$.

67. u^3; $\frac{1}{4}(x^2+3x-2)^4$.

68. $-\frac{1}{2}u^{\frac{1}{2}}$; $-\frac{1}{3}(3-x^2)^{\frac{3}{2}}$.

69. $\frac{1}{2a}u^{\frac{1}{2}}$; $\frac{1}{3a}(ax^2+b)^{\frac{3}{2}}$.

70. $\frac{1}{2u^{\frac{1}{2}}}$; $\sqrt{(x^2+1)}$.

71. $\frac{1}{4u^{\frac{1}{2}}}$; $\frac{1}{2}\sqrt{(2x^2-4x+1)}$.

72. $\frac{1}{u^{\frac{1}{2}}}$; $2\sqrt{(ax^2+bx+c)}$.

73. $\frac{u^{\frac{1}{2}}}{3a}$; $\frac{2(ax^3+b)^{\frac{3}{2}}}{9a}$.

74. u^n; $\frac{1}{n+1}(ax^2+bx+c)^{n+1}$.

75. u^n; $\frac{1}{n+1}[f(x)]^{n+1}$.

Exercises. VI. Page 63.

1. x. **2.** $\frac{1}{2}x^2$. **3.** $\frac{1}{2}x^2+x$. **4.** x^2+x. **5.** $\frac{1}{4}x^2+\frac{1}{2}x$.

6. $\frac{1}{2}ax^2+bx$. **7.** $\frac{5}{3}x^3-\frac{3}{2}x^2+4x$. **8.** $\frac{1}{3}x^3-\frac{3}{2}x^2+2x$.

9. $\frac{2}{3}x^3-\frac{5}{2}x^2+2x$. **10.** $\frac{1}{3}ax^3+\frac{1}{2}bx^2+cx$.

11. $\frac{1}{3}aAx^3+\frac{1}{2}(aB+Ab)x^2+bBx$. **12.** $\frac{1}{2}x^2-\frac{1}{4}x^4$.

13. $3x+\frac{5}{2}x^2+\frac{1}{3}x^3-\frac{3}{4}x^4$. **14.** $\frac{1}{4}ax^4+\frac{1}{3}bx^3+\frac{1}{2}cx^2+dx$.

15. $\frac{2}{3}\sqrt{x^3}$. **16.** $2\sqrt{x}$. **17.** $-\frac{1}{x}$. **18.** $-\frac{1}{0{\cdot}41x^{0{\cdot}41}}$.

19. $\log(x+1)$. **20.** $-\frac{1}{x+1}$. **21.** $x+2\log x$. **22.** $\log x-\frac{2}{x}$.

23. $3x-2\log x-\frac{1}{x}$. **24.** $ax+b\log x-\frac{c}{x}$. **25.** $\frac{1}{8}(2x+1)^4$.

26. $-\frac{1}{4(2x+1)^2}$. **27.** $\frac{1}{3}\sqrt{(2x+3)^3}$. **28.** $\sqrt{(2x+3)}$.

29. $\frac{2}{3a}\sqrt{(ax+b)^3}$. **30.** $\frac{2}{a}\sqrt{(ax+b)}$. **31.** $\frac{1}{2}\log(2x-3)$

32. $-\frac{1}{2}\log(3-2x)$. **33.** $x+2\log(x-1)$. **34.** $x+\frac{3}{2}\log(2x-3)$.

35. $\frac{1}{3}x^3+2x^2+5\log(x-1)$.
36. $\frac{2}{3}x^3-\frac{7}{2}x^2+14x-26\log(x+2)$.
37. $\frac{1}{2}\log\frac{x-1}{x+1}$.
38. $\frac{1}{4}\log\frac{x-2}{x+2}$.
39. $\frac{1}{2\sqrt{3}}\log\frac{x-\sqrt{3}}{x+\sqrt{3}}$.
40. $\frac{1}{12}\log\frac{2x-3}{2x+3}$.
41. $\frac{\sqrt{21}}{42}\log\frac{3x-\sqrt{21}}{3x+\sqrt{21}}$.
42. $\frac{1}{2}\log(x^2-4)$.
43. $\frac{1}{2}\log(x^2-a^2)$.
44. $4\log(x-3)-3\log(x-2)$.
45. $\log(x-1)+\frac{1}{2}\log(2x+3)$.
46. $\frac{1}{a-b}\log\frac{x-a}{x-b}$.
47. $\frac{1}{a-b}[a\log(x-a)-b\log(x-b)]$.
48. $\log\frac{(x-3)(x-2)}{(x-1)^2}$.
49. $\log\frac{x-1}{x}+\frac{1}{x}$.
50. $\log\frac{x+2}{x-1}-\frac{1}{x-1}$.
51. $y=6x-\frac{3}{2}x^2$.
52. $y=\frac{1}{2}ax^2+bx+c$.
53. $y=\frac{1}{2}x^2-\frac{1}{3}x^3+\frac{5}{6}$.
54. $y=\frac{1}{2}x^2+\frac{1}{x}+\frac{1}{2}$.
55. $y=\frac{1}{6}x^3+\frac{1}{2}x^2+x$.
56. $x=Vt\cos\alpha$; $y=Vt\sin\alpha-\frac{1}{2}gt^2$.
57. $x=Vt\cos\alpha$; $y=Vt\sin\alpha-\frac{1}{2}gt^2$.
58. $E=\frac{1}{2}km(a^2-x^2)$.
59. $E=\frac{km}{x}$.
60. $EIy=\frac{1}{24}W(3Lx^2-2x^3)$; $\frac{1}{24}\frac{WL^3}{EI}$.
61. $EIy=\frac{1}{48}W\{3L^2x-4x^3\}$; $\frac{WL^3}{48EI}$.
62. $EIy=\frac{w}{24}(L^3x-2Lx^3+x^4)$; $\frac{5wL^4}{384EI}$.
63. $EIy=\frac{w}{24}x^2(L-x)^2$; $\frac{wL^4}{384EI}$.

Exercises. VII. Page 67.

1. $\frac{1}{2}\log(x^2+a^2)$.
2. $\frac{1}{2}\log(x^2+2ax+b^2)$.
3. $\frac{1}{2}\log(ax^2+2bx+c)$.
4. $-\frac{1}{3}\sqrt{(3-x^2)^3}$.
5. $-\sqrt{(3-x^2)}$.
6. $\frac{1}{3a}\sqrt{(ax^2+b)^3}$.
7. $\frac{1}{3}\sqrt{(x^2-2ax+b^2)^3}$.
8. $\sqrt{(ax^2+2bx+c)}$.
9. $\frac{2}{9}\sqrt{(x^3-1)^3}$.
10. $-\frac{1}{2(x^2+a^2)}$.
11. $-\frac{1}{x^2-3x+8}$.
12. $-\sqrt{(3x^2-14x+10)}$.
13. $\frac{2}{105}(15x^2-12x+8)\sqrt{(x+1)^3}$, *or*, $\frac{2}{105}(15x^3+3x^2-4x+8)\sqrt{(x+1)}$.
14. $\frac{2}{15}(6x+13)\sqrt{(x+3)^3}$, *or*, $\frac{2}{15}(6x^2+31x+39)\sqrt{(x+3)}$.
15. $\frac{2}{35}(5x^3+6x^2+8x+16)\sqrt{(x-1)}$.
16. $\frac{\sqrt{(ax^4+b)^3}}{6a}$.
17. $\frac{(ax^n+b)^{m+1}}{(m+1)na}$.
18. $\frac{1}{n+1}[f(x)]^{n+1}$.

19. $\log f(x)$.

20. $\frac{1}{15}(3x^2-2)\sqrt{(x^2+1)^3}$.

21. $\dfrac{2(x+2)}{\sqrt{(x+1)}}$.

22. $\dfrac{2(ax+2b)}{a^2\sqrt{(ax+b)}}$.

23. $\dfrac{x}{\sqrt{(x^2+1)}}$.

24. $\dfrac{x}{a^2\sqrt{(x^2+a^2)}}$.

25. $\dfrac{x+2}{\sqrt{(x^2+4x+5)}}$.

26. $\dfrac{x+a}{(b-a^2)\sqrt{(x^2+2ax+b)}}$.

Exercises. VIII. Page 75.

1. 4. **2.** 36. **3.** $17\frac{1}{2}$. **4.** 21. **5.** $-63\frac{3}{4}$.

6. 5. **7.** -190. **8.** 0. **9.** $\frac{2}{3}$. **10.** $\frac{4}{9}$.

11. $\frac{26}{81}$. **12.** $562\frac{1}{2}$. **13.** $\frac{5}{11}$. **14.** $\dfrac{32\sqrt{2}}{3}$.

15. $\dfrac{32\sqrt{2}}{3}$. **16.** 2. **17.** 2. **18.** $3\frac{3}{14}$.

19. $2\log 2 = 1{\cdot}3863$. **20.** $\log\dfrac{b^2}{a^2}$. **21.** $\log 4 = 1{\cdot}3863$.

22. $\frac{1}{2}\log 5 = 0{\cdot}8047$. **23.** $\frac{1}{4}\log\frac{15}{7} = 0{\cdot}1905$. **24.** $\frac{1}{2}\log 3 = 0{\cdot}5493$.

25. $\dfrac{5}{2}\left[\dfrac{1}{a^{0\cdot4}} - \dfrac{1}{b^{0\cdot4}}\right]$. **26.** $32\frac{2}{3}$. **27.** $2\frac{2}{3}$ **28.** $\frac{1}{3}a^3$. **29.** a.

30. $\dfrac{1}{2}\log\dfrac{a^2+b^2}{a^2}$. **31.** $\dfrac{458\sqrt{2}}{35}$. **32.** $2\frac{2}{3}$. **33.** $\frac{4}{45}$.

34. $98\frac{46}{105}$. **35.** $58\frac{1}{3}$. **36.** $\frac{1}{4}$. **37.** $\frac{2}{3}$.

38. $\frac{2}{3}ah^3 + 2ch$. $\quad a = \dfrac{y_1+y_3-2y_2}{2h^2}$; $\quad b = \dfrac{y_3-y_1}{2h}$; $\quad c = y_2$.

41. $c^2\log\dfrac{b}{a}$. **42.** $\dfrac{k}{n-1}\left\{\dfrac{1}{x_1^{n-1}} - \dfrac{1}{x_2^{n-1}}\right\}$. **43.** $\dfrac{1}{(n+1)c^{n-1}}\{x_2^{n+1} - x_1^{n+1}\}$.

44. $c^3\left\{1-\dfrac{1}{b}\right\}$; c^3. **45.** πab.

47. (i) $\dfrac{\pi}{2}$; (ii) π; (iii) $\dfrac{\pi(b-a)^2}{8}$; (iv) $\dfrac{\pi(b+a)(b-a)^2}{16}$. **50.** $\dfrac{h^3}{12a}$.

Exercises. X. Page 100.

7. (i) $\frac{2}{3}h$; (iii) $\frac{3}{4}h$. **11.** $\frac{5}{4}Ma^2$. **12.** $\frac{3}{2}Ma^2$.

13. $\frac{7}{5}Ma^2$. **14.** $M(\frac{2}{5}a^2+c^2)$. **15.** $M\left(\dfrac{a^2+4b^2}{12}\right)$.

16. (i) $M\dfrac{a^2}{2}$; (ii) $M\left[\dfrac{a^2}{4}+\dfrac{h^2}{3}\right]$. **17.** $M\dfrac{a^2+b^2}{2}$.

18. (i) $\frac{3}{10}Ma^2$; (ii) $\frac{3}{20}M(a^2+4h^2)$. **19.** (i) $\frac{1}{6}Mh^2$; (ii) $\frac{1}{2}Mh^2$.

21. 42610 foot-pounds; 72°. **22.** 7980 foot-pounds; 93° F.

23. $p_1 v_1\left(1-\frac{\tau_2}{\tau_1}\right)\log\frac{v_2}{v_1}$. Note that, by § 40, example 1 (6), the *total* work done during adiabatic expansion and compression is zero; also that, by § 40, (ii) (iii), $v_3/v_4 = v_2/v_1$.

Exercises. XI. Page 107.

1. $2\cos 2x$.
2. $-2\sin 2x$.
3. $2\cos(2x+5)$.
4. $-2\sin(2x+5)$.
5. $-\cos(3-x)$.
6. $\sin(3-x)$.
7. $\frac{1}{2}\cos\left(\frac{1}{2}x+\frac{1}{3}\pi\right)$.
8. $-\frac{1}{2}\sin\left(\frac{1}{2}x+\frac{1}{3}\pi\right)$.
9. $-2\cos(3-2x)$.
10. $2\sin(3-2x)$.
11. $5\cos 5\left(x-\frac{1}{2}\pi\right)$.
12. $-5\sin 5\left(x-\frac{1}{2}\pi\right)$.
13. $\frac{2\pi}{3}\cos\frac{2\pi}{3}(x+2)$.
14. $-\frac{2\pi}{3}\sin\frac{2\pi}{3}(x+2)$.
15. $\frac{2\pi}{a}\cos\frac{2\pi}{a}(x+b)$.
16. $-\frac{2\pi}{a}\sin\frac{2\pi}{a}(x+b)$.
17. $2\cos 2x\cos x-\sin 2x\sin x$.
18. $\cos 2x\cos x-2\sin 2x\sin x$.
19. $m\cos mx\cos nx-n\sin mx\sin nx$
20. $m\cos mx\sin nx+n\sin mx\cos nx$.
21. $3\sec^2(3x-4)$.
22. $-3\operatorname{cosec}^2(3x-4)$.
23. $-2\operatorname{cosec}(2x-3)\cot(2x-3)$.
24. $3\sec(3x-2)\tan(3x-2)$.
25. $\sin x+x\cos x$.
26. $\cos x-x\sin x$.
27. $2x\sin x+x^2\cos x$.
28. $2x\cos x-x^2\sin x$.
29. $x\cos x$.
30. $x\sin x$.
31. $\cos^2 x$.
32. $\sin^2 x$.
33. $\cos^3 x$.
34. $\sin^3 x$.
35. $\tan^2 x$.
36. $\tan x+x\sec^2 x$.
37. $\frac{\cos x}{x}-\frac{\sin x}{x^2}$.
38. $-\frac{\sin x}{x}-\frac{\cos x}{x^2}$.
39. $-\frac{2\cos x}{(1+\sin x)^2}$.
40. $\frac{2\sin x}{(1+\cos x)^2}$.
41. $\frac{2}{1+\sin 2x}$.
42. $-\frac{1}{4}\operatorname{cosec}\frac{x}{2}\cot\frac{x}{2}$.
43. $x^2\sin x$.
44. $4x\sin 2x$.

Exercises. XII. Page 110.

1. $-3\sin(6x-4)$.
2. $12\sin^2(4x-1)\cos(4x-1)$.
3. $-4a\cos^3(ax+b)\sin(ax+b)$.
4. $na\sin^{n-1}(ax+b)\cos(ax+b)$.
5. $\frac{\cos x}{2\sqrt{\sin x}}$.
6. $\frac{-\sin 2x}{\sqrt{\cos 2x}}$.
7. $2\sin x\cos^4 x-3\sin^3 x\cos^2 x$.
8. $m\sin^{m-1}x\cos^{n+1}x-n\sin^{m+1}x\cos^{n-1}x$.
9. $\frac{-\cos x}{\sin^2 x}$.
10. $\frac{-2\cos x}{\sin^3 x}$.
11. $\frac{1+\sin^2 x}{\cos^3 x}$.
12. $\frac{\sin^{m-1}x(m\cos^2 x+n\sin^2 x)}{\cos^{n+1}x}$.
13. $2\tan x\sec^2 x$.
14. $2\sec^2 x\tan x$.
15. $2a\tan(ax+b)\sec^2(ax+b)$.
16. $2x\tan x(\tan x+x\sec^2 x)$.

17. $2x\sin^2x + x^2\sin 2x$.

18. $2x\cos^3x - 3x^2\cos^2x\sin x$.

19. $nx^{n-1}\sin^{n-1}x(\sin x + x\cos x)$.

20. $nx^{n-1}\cos^{n-1}x(\cos x - x\sin x)$.

21. Min. when $x=0$, π; Max. when $x=\pi/2$, $3\pi/2$; Inflexion when $x=\pi/4$, $3\pi/4$, $5\pi/4$, $7\pi/4$. The period is π, but in this and the following examples the values in the range from $x=0$ to $x=2\pi$ are given.

22. Max. when $x=\pi/2$; Min. when $x=3\pi/2$; Inflexion when $x=0$, $0{\cdot}955$, $2{\cdot}186$, π, $4{\cdot}097$, $5{\cdot}328$.

23. Min. when $x=0$, π; Max. when $x=\pi/2$, $3\pi/2$; Inflexion when $x=\pi/3$, $2\pi/3$, $4\pi/3$, $5\pi/3$.

24. Max. when $x=0$, π; Min. when $x=\pi/2$, $3\pi/2$; Inflexion when $x=\pi/4$, $3\pi/4$, $5\pi/4$, $7\pi/4$.

25. Max. when $x=0$, 2π; Min. when $x=\pi$; Inflexion when $x=0{\cdot}616$, $\pi/2$, $2{\cdot}526$, $3{\cdot}757$, $3\pi/2$, $5{\cdot}668$.

26. Max. when $x=0$, π; Min. when $x=\pi/2$, $3\pi/2$; Inflexion when $x=\pi/6$, $5\pi/6$, $7\pi/6$, $11\pi/6$.

27. Min. when $x=0$, π; no point of inflexion.

28. Inflexion when $x=0$, π; no turning value.

29. $2\cos 2x$.

30. $-2\cos 2x$.

31. $2\sec^2x(\sec^2x + 2\tan^2x)$, *or* $2(1+\tan^2x)(1+3\tan^2x)$.

32. $-(34\sin 5x\cos 3x + 30\cos 5x\sin 3x)$, *or* $-2(\sin 2x + 16\sin 8x)$.

33. $-(34\cos 5x\sin 3x + 30\sin 5x\cos 3x)$, *or* $2(\sin 2x - 16\sin 8x)$.

34. $2mn\cos mx\cos nx - (m^2+n^2)\sin mx\sin nx$,
or $\frac{1}{2}(m+n)^2\cos(m+n)x - \frac{1}{2}(m-n)^2\cos(m-n)x$.

35. $-2mn\cos mx\sin nx - (m^2+n^2)\sin mx\cos nx$,
or $-\frac{1}{2}(m+n)^2\sin(m+n)x - \frac{1}{2}(m-n)^2\sin(m-n)x$.

36. $a^n\cos\left(ax+b+\frac{n\pi}{2}\right)$.

37. $2^{n-1}\sin\left(2x+\overline{n-1}\,\frac{\pi}{2}\right)$, *or* $-2^{n-1}\cos\left(2x+\frac{n\pi}{2}\right)$.

38. $\frac{3}{4}\sin\left(x+\frac{n\pi}{2}\right) - \frac{3^n}{4}\sin\left(3x+\frac{n\pi}{2}\right)$.

39. $-2^{n-1}\sin\left(2x+\overline{n-1}\,\frac{\pi}{2}\right)$, *or* $2^{n-1}\cos\left(2x+\frac{n\pi}{2}\right)$.

40. $\frac{3^n}{4}\cos\left(3x+\frac{n\pi}{2}\right) + \frac{3}{4}\cos\left(x+\frac{n\pi}{2}\right)$.

42. $54°\ 44'$; $125°\ 16'$.

48. 1.

49. $a^4\cos^4u$.

50. 1.

51. $a^2\cos^2u$.

52. 2.

53. $\frac{1}{2}(b-a)^2\sin^2 2u$.

54. $2a^3\sin^6u$.

55. $1/b$.

Exercises. XIII. Page 117.

1. $0{\cdot}5403023$.

6. $x^2 - \frac{1}{3}x^4 + \ldots + \frac{1}{2}(-1)^{n-1}\frac{(2x)^{2n}}{(2n)!} + \ldots$.

7. (i) $1 - x^2 + \frac{1}{3}x^4 - \ldots + \frac{1}{2}(-1)^n \frac{(2x)^{2n}}{(2n)!} + \ldots$;

(ii) $x^3 - \frac{x^5}{2} + \frac{13x^7}{120} - \ldots + (-1)^n \left(\frac{3}{4} - \frac{3^{2n+1}}{4}\right) \frac{x^{2n+1}}{(2n+1)!} + \ldots$;

(iii) $1 - 2x^2 + \frac{7x^4}{8} - \ldots + (-1)^n \left(\frac{3}{4} + \frac{3^{2n}}{4}\right) \frac{x^{2n}}{(2n)!} + \ldots$;

(iv) $3x - 6x^3 + \frac{22x^5}{5} - \ldots + \frac{1}{2}(-1)^n (4^{2n+1} + 2^{2n+1}) \frac{x^{2n+1}}{(2n+1)!} + \ldots$.

8. $-\frac{1}{6}$. **9.** 2. **10.** 2. **11.** 1. **12.** 128/81.

Exercises. XIV. Page 122.

1. $-\frac{1}{3}\cos 3x$. **2.** $\cos(1-x)$. **3.** $-\sin(1-x)$.

4. $-\frac{a}{2\pi}\cos\frac{2\pi}{a}(x+b)$. **5.** $\frac{a}{2\pi}\sin\frac{2\pi}{a}(x+b)$. **6.** $\frac{x}{2} - \frac{1}{4n}\sin 2(nx+a)$.

7. $\frac{x}{2} + \frac{1}{4n}\sin 2(nx+a)$. **8.** $\tan x - x$. **9.** $\frac{1}{3}\sin^3 x$.

10. $-\frac{1}{3}\cos^3 x$. **11.** $\frac{1}{2}\tan^2 x$. **12.** $-\frac{1}{2}\cot^2 x$.

13. $\frac{\pi}{2}$. **14.** $\frac{T}{2}$. **15.** $\frac{T}{2}$. **16.** $\frac{\pi}{8}$. **17.** $\frac{\pi}{n}$.

18. $\frac{2}{3}$. **19.** $\frac{3\pi}{16}$ **20.** 0. **21.** 0.

29. $\frac{3\pi}{16}a^4$. **30.** $\frac{\pi}{16}a^4$. **31.** $\frac{5\pi}{32}a^6$. **32.** $\frac{5\pi}{16}a^3$.

33. $\frac{\pi}{2}a^2$. **34.** πa. **35.** π. **36.** $\frac{\pi}{8}(b-a)^2$.

37. $\frac{1}{4} + \frac{\pi}{8}$. **38.** $2\pi a^2\left\{1 - e^2 + \frac{\sqrt{1-e^2}}{e}\sin^{-1}e\right\}$; $4\pi a^2$. **39.** $\frac{\pi a^4}{8b^2}$.

40. (ii) $\frac{1}{5}\sin^5 x - \frac{2}{7}\sin^7 x + \frac{1}{9}\sin^9 x$; (iii) $\sin x - \frac{1}{3}\sin^3 x$;

(iv) $\sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x$.

41. (i) $\frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x$; (ii) $\frac{2}{7}\cos^7 x - \frac{1}{5}\cos^5 x - \frac{1}{9}\cos^9 x$;

(iii) $\frac{1}{3}\cos^3 x - \cos x$; (iv) $\frac{2}{3}\cos^3 x - \cos x - \frac{1}{5}\cos^5 x$.

42. (ii) $\frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x + \tan x - x$;

(iii) $\frac{1}{7}\tan^7 x - \frac{1}{5}\tan^5 x + \frac{1}{3}\tan^3 x - \tan x + x$;

(iv) $\frac{1}{9}\tan^9 x - \frac{1}{7}\tan^7 x + \frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x + \tan x - x$.

Exercises. XV. Page 128.

4. (i) $\frac{I}{\sqrt{2}}$; (ii) $\sqrt{\left(\frac{I_1^2 + I_2^2}{2}\right)}$;

(iii) $\sqrt{\left(\frac{I_1^2 + I_2^2 + I_3^2}{2}\right)}$; (iv) $\sqrt{\left\{\frac{I_1^2 + I_2^2 + \ldots + I_n^2}{2}\right\}}$.

5. (i) $\frac{1}{2}IE\cos\beta$; (ii) $\frac{1}{2}I_1E_1\cos(\alpha_1-\beta_1)+\frac{1}{2}I_2E_2\cos(\alpha_2-\beta_2)$.

6. (i) A_0; (ii) A_r; (iii) B_r. **7.** $(-1)^{r-1}\frac{\pi}{r}$. **8.** $-\frac{2\pi}{r}$.

9. $\frac{\cos r\pi-1}{r^2}$. **10.** 0. **11.** $(-1)^r\frac{2\pi}{r^2}$.

12. $-\frac{\pi^2\cos r\pi}{r}+\frac{2(\cos r\pi-1)}{r^3}$. **13.** $(-1)^r\frac{a\sin a\pi}{a^2-r^2}$. **14.** $(-1)^r\frac{r\sin a\pi}{a^2-r^2}$.

15. $\frac{35\pi}{256}$. **16.** $\frac{128}{315}$. **17.** $\frac{32}{35}$. **18.** $\frac{35\pi}{128}$.

19. $\frac{3\pi}{4}$. **20.** $\frac{35\pi}{32}$. **21.** $\frac{2}{35}$. **22.** $\frac{5\pi}{4096}$.

23. $\frac{16}{315}$. **24.** $\frac{63\pi}{2^{17}}$. **25.** $\frac{\pi a^4}{16}$. **26.** $\frac{\pi a^3}{2}$.

27. $\frac{5\pi a^4}{8}$. **28.** $\frac{5\pi a^8}{256}$. **29.** πa^2. **30.** $\frac{8\pi a^3}{5}$.

Exercises. XVI. Page 152.

1. $26{\cdot}42+24{\cdot}57\cos x+1{\cdot}08\cos 2x-20{\cdot}33\cos 3x+1{\cdot}58\cos 4x$
$+1{\cdot}76\cos 5x-3{\cdot}08\cos 6x+15{\cdot}19\sin x-13{\cdot}42\sin 2x$
$+20{\cdot}17\sin 3x-3{\cdot}03\sin 4x-5{\cdot}02\sin 5x$.

2. $7{\cdot}42+29{\cdot}70\cos x-0{\cdot}08\cos 2x-13{\cdot}17\cos 3x-6{\cdot}92\cos 4x-0{\cdot}03\cos 5x$
$-4{\cdot}92\cos 6x-14{\cdot}32\sin x-13{\cdot}13\sin 2x+18{\cdot}33\sin 3x$
$-6{\cdot}21\sin 4x+4{\cdot}15\sin 5x$.

3. $0{\cdot}075+4{\cdot}54\cos x+1{\cdot}27\cos 2x+0{\cdot}02\cos 3x-0{\cdot}10\cos 4x+0{\cdot}09\cos 5x$
$+0{\cdot}11\cos 6x+7{\cdot}47\sin x+1{\cdot}79\sin 2x+0{\cdot}10\sin 3x$
$+0{\cdot}12\sin 4x+0{\cdot}03\sin 5x$.

4. $1{\cdot}50+18{\cdot}24\cos x+1{\cdot}00\cos 2x-1{\cdot}37\cos 3x-0{\cdot}20\cos 4x+0{\cdot}23\cos 5x$
$+3{\cdot}69\sin x+0{\cdot}06\sin 2x-0{\cdot}40\sin 3x+0{\cdot}40\sin 4x+0{\cdot}11\sin 5x$.

7. $\frac{\pi}{2}-\frac{4}{\pi}\left[\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+\ldots+\frac{\cos(2n+1)x}{(2n+1)^2}+\ldots\right]$.

8. $\sin(-x)$, *or* $-\sin x$.

9. $\frac{1}{2}+\frac{2}{\pi}\left[\frac{\cos x}{1}-\frac{\cos 3x}{3}+\frac{\cos 5x}{5}-\frac{\cos 7x}{7}+\ldots\right]$; 1, $\frac{1}{2}$, 0.

10. $\frac{2}{\pi}\left[\cos x-\frac{1}{3}\cos 3x+\frac{1}{5}\cos 5x-\frac{1}{7}\cos 7x+\ldots\right]$
$+\frac{2}{\pi}\left[\sin x+\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x+\frac{1}{7}\sin 7x+\ldots\right]$; $-\frac{1}{2}$, $-\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{2}$, $-\frac{1}{2}$.

11. $\frac{2\sqrt{3}}{\pi}\left[\sin x-\frac{1}{5^2}\sin 5x+\frac{1}{7^2}\sin 7x-\frac{1}{11^2}\sin 11x+\ldots\right]$; $B_r=\frac{4}{\pi r^2}\sin\frac{r\pi}{2}\cos\frac{r\pi}{6}$.
Series represents x from $x=0$ to $x=-\frac{\pi}{3}$; $-\frac{\pi}{3}$ from $x=-\frac{\pi}{3}$ to $x=-\frac{2\pi}{3}$; and $-(\pi+x)$ from $x=-\frac{2\pi}{3}$ to $x=-\pi$.

12. $\frac{4}{\pi}\left\{\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \dots\right\}$.

13. $\frac{\pi^2}{3} - 4\left\{\cos x - \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x - \frac{1}{4^2}\cos 4x + \dots\right\}$;

$\frac{2}{\pi}\left\{\left(\frac{\pi^2}{1} - \frac{4}{1^3}\right)\sin x - \frac{\pi^2}{2}\sin 2x + \left(\frac{\pi^2}{3} - \frac{4}{3^3}\right)\sin 3x - \frac{\pi^2}{4}\sin 4x + \dots\right]$.

15. $\frac{\pi}{\sin a\pi}$. 17. $\frac{\pi}{2}\sec\frac{a\pi}{2}$. 18. $\frac{\pi}{2\sqrt{2}}$; put $x = \frac{\pi}{4}$.

Exercises. XVII. Page 159.

1. $\frac{2}{\sqrt{(1-4x^2)}}$. 2. $-\frac{1}{\sqrt{(2x-x^2)}}$. 3. $\frac{1}{\sqrt{(2+x-x^2)}}$.

4. $\frac{1}{\sqrt{(2x-x^2)}}$. 5. $\frac{a}{\sqrt{\{1-(ax+b)^2\}}}$. 6. $\frac{-1}{2-2x+x^2}$.

7. $\frac{3}{5+2x+2x^2}$. 8. $\sin^{-1}x + \frac{x}{\sqrt{(1-x^2)}}$. 9. $\sin^{-1}x$.

10. $x\sin^{-1}x$. 11. $\sqrt{(3+2x-x^2)}$. 12. $\tan^{-1}x + \frac{x}{1+x^2}$.

13. $2x\tan^{-1}x$. 14. $\sin^{-1}\left(\frac{x}{\sqrt{3}}\right)$. 15. $\frac{1}{2}\sin^{-1}\left(\frac{2x}{3}\right)$.

16. $\frac{1}{\sqrt{3}}\sin^{-1}\left(\frac{x\sqrt{3}}{\sqrt{7}}\right)$. 17. $\frac{1}{2}x\sqrt{(3-x^2)} + \frac{3}{2}\sin^{-1}\left(\frac{x}{\sqrt{3}}\right)$.

18. $\frac{1}{2}x\sqrt{(7-3x^2)} + \frac{7\sqrt{3}}{6}\sin^{-1}\left(\frac{x\sqrt{3}}{\sqrt{7}}\right)$. 19. $\frac{1}{2}x\sqrt{(b^2-a^2x^2)} + \frac{b^2}{2a}\sin^{-1}\left(\frac{ax}{b}\right)$.

20. $\sin^{-1}\left(\frac{x-3}{3}\right)$. 21. $\frac{x-3}{2}\sqrt{(6x-x^2)} + \frac{9}{2}\sin^{-1}\left(\frac{x-3}{3}\right)$.

22. $\sin^{-1}\left(\frac{2x-a}{a}\right)$. 23. $\frac{2x-a}{4}\sqrt{(ax-x^2)} + \frac{a^2}{8}\sin^{-1}\left(\frac{2x-a}{a}\right)$.

24. $\frac{1}{2}\sin^{-1}\left(\frac{2x-1}{2}\right)$. 25. $\frac{(2x-1)}{4}\sqrt{(3+4x-4x^2)} + \sin^{-1}\left(\frac{2x-1}{2}\right)$.

26. $\frac{3x-4}{6}\sqrt{(24x-9x^2-7)} + \frac{3}{2}\sin^{-1}\left(\frac{3x-4}{3}\right)$. 27. $\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)$.

28. $\frac{1}{ab}\tan^{-1}\left(\frac{bx}{a}\right)$. 29. $\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$.

30. $\frac{1}{9}\tan^{-1}\left(\frac{3x+4}{3}\right)$. 31. $\frac{\pi}{6}$. 32. $\frac{\pi}{2}$.

33. $\frac{\pi}{4a}$. 34. $\frac{\pi}{2}$. 35. π. 36. π.

Exercises. XVIII. Page 167.

1. $-\frac{2a}{x^2-a^2}$; $\frac{1}{2a}\log\left(\frac{x-a}{x+a}\right)$. 2. $\frac{2a}{a^2-x^2}$; $\frac{1}{2a}\log\left(\frac{a+x}{a-x}\right)$.

3. $\dfrac{\sqrt{a}}{(a-x)\sqrt{x}}$. **4.** $\dfrac{1}{2\sqrt{(x^2-a^2)}}$. **5.** $\dfrac{1}{\sqrt{(x^2-ax)}}$. **6.** $\dfrac{2a(a^2-x^2)}{x^4+a^2x^2+a^4}$.

7. $\cot x$. **8.** $-\tan x$. **9.** $2\sec x$. **10.** $-2\operatorname{cosec} x$.

11. $\dfrac{\cos x}{1+\sin x}$. **12.** $\dfrac{1+\cos x}{x+\sin x}$. **13.** $\dfrac{2\cos x}{\sin x+\cos x}$.

14. $\dfrac{5(\sin x+\cos x)}{\sin x+2\cos x}$. **15.** $\tan^{-1}x$. **16.** $\dfrac{x^2+x+1}{x^2-x+1}$.

17. $\sqrt{(x^2+2x+3)}$. **18.** $1+\log x$. **19.** $x^{n-1}+nx^{n-1}\log x$.

20. $\dfrac{1}{x^3(x-1)}$. **21.** $e^x(x+1)$. **22.** $e^{-x}(1-x)$.

23. $x^{n-1}e^x(x+n)$. **24.** $x^{n-1}e^{-x}(n-x)$. **25.** $10^x\log_e 10$.

26. $-10^{-x}\log_e 10$. **27.** $e^x(\sin x+\cos x)$. **28.** $e^x(\cos x-\sin x)$.

29. $e^{-3x}\{-3\sin(4x+5)+4\cos(4x+5)\}$.

30. $-e^{-3x}\{3\cos(4x+5)+4\sin(4x+5)\}$.

31. $e^{ax}\{a\sin(bx+c)+b\cos(bx+c)\}$. **32.** $e^{ax}\{a\cos(bx+c)-b\sin(bx+c)\}$.

33. $\frac{1}{2}\{e^{\frac{x}{a}}-e^{-\frac{x}{a}}\}$. **34.** $\frac{1}{2}(e^{\frac{x}{a}}+e^{-\frac{x}{a}})$. **35.** a^ne^{ax}.

36. $-\dfrac{1}{x^2}$; $\dfrac{2}{x^3}$; $\dfrac{(-1)^{n-1}(n-1)!}{x^n}$. **42.** $\log(x+\sqrt{x^2-1})$.

43. $\dfrac{1}{\sqrt{3}}\log\{x+\sqrt{(x^2+\frac{1}{3})}\}$. **44.** $\log\{x+1+\sqrt{(x^2+2x+2)}\}$.

45. $\dfrac{1}{\sqrt{3}}\log\left\{x+\dfrac{2}{3}+\sqrt{\left(x^2+\dfrac{4x}{3}-\dfrac{5}{3}\right)}\right\}$. **46.** $\log\{x+a+\sqrt{(x^2+2ax+b^2)}\}$.

47. $\frac{1}{4}(2x-3)\sqrt{\{(x-1)(x-2)\}}-\frac{1}{8}\log[x-\frac{3}{2}+\sqrt{\{(x-1)(x-2)\}}]$.

48. $\frac{1}{2}(x+a)\sqrt{(x^2+2ax+b^2)}+\frac{1}{2}(b^2-a^2)\log\{x+a+\sqrt{(x^2+2ax+b^2)}\}$.

49. $\frac{1}{2}x^2\log x-\frac{1}{4}x^2$. **50.** $\frac{1}{3}x^3\log x-\frac{1}{9}x^3$.

51. $\dfrac{1}{n+1}x^{n+1}\log x-\dfrac{1}{(n+1)^2}x^{n+1}$. **52.** $e^x(x-1)$.

53. $e^x(x^2-2x+2)$. **54.** $e^x(x^3-3x^2+6x-6)$. **55.** $-e^{-x}(x+1)$.

56. $-e^{-x}(x^2+2x+2)$. **57.** $-e^{-x}(x^3+3x^2+6x+6)$.

58. $T_2=T_1e^{-\mu\pi}$; (i) $T_2=T_1e^{-2\mu\pi}$; (ii) $T_2=T_1e^{-4\mu\pi}$.

59.

	Napierian Logarithms.	Common Logarithms.
2.	0·69315	0·3010
3.	1·09861	0·4771
4.	1·38629	0·6021
5.	1·60944	0·6990
6.	1·79176	0·7782
7.	1·94591	0·8451
8.	2·07944	0·9031
9.	2·19722	0·9542
10.	2·30259	1

Exercises. XIX. Page 173.

1. $\frac{1}{2}e^x(\sin x - \cos x)$.
2. $\frac{1}{2}e^x(\sin x + \cos x)$.
3. $-\frac{1}{2}e^{-x}(\sin x + \cos x)$.
4. $\frac{1}{2}e^{-x}(\sin x - \cos x)$.
5. $-\frac{1}{25}e^{-3x}(3\sin 4x + 4\cos 4x)$.
6. $\frac{1}{25}e^{-4x}(3\sin 3x - 4\cos 3x)$.
7. $\frac{1}{10}e^x(5 - 2\sin 2x - \cos 2x)$.
8. $\frac{1}{10}e^{-x}(2\sin 2x - \cos 2x - 5)$.
9. $\frac{1}{5}$.
10. $\frac{1}{25}\{3\cos 5 - 4\sin 5\} = 0{\cdot}1875$.
11. $\frac{n}{k^2+n^2}$.
12. $\frac{k}{k^2+n^2}$.
13. $\frac{k\sin\alpha + n\cos\alpha}{k^2+n^2}$.
14. $\frac{\pi}{4}$.
15. $\frac{\pi}{2}$.
16. 1.
17. 1.
18. $\frac{1}{4}$.
19. $\frac{\pi}{4a^3}$.
20. (i) $\frac{\pi}{4}$; (ii) π.
21. $\frac{\pi}{2ab}$.
27. $x = e^{-\frac{1}{2}kt}\left\{a\cos nt + \frac{2V + ka}{2n}\sin nt\right\}$.

Exercises. XX. Page 180.

1. $x_1x + y_1y = R^2$; $y_1x - x_1y = 0$.
2. $y_1y = 2a(x + x_1)$; $y_1(x - x_1) + 2a(y - y_1) = 0$.
   $y = x/t + at$; $y = -tx + 2at + at^3$.
4. $\frac{x_1x}{a^2} + \frac{y_1y}{b^2} = 1$; $\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$.
   $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$; $\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$.
6. P lies between K and L.
9. Subtangent $= a\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right) \div \left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right)$;
   Subnormal $= \frac{1}{4}a\left(e^{\frac{2x}{a}} - e^{-\frac{2x}{a}}\right)$; Normal $= y^2/a$.
10. Subtangent $= b\tan\frac{x}{b}$; Subnormal $= \frac{a^2}{2b}\sin\frac{2x}{b}$.
11. (i) 2; (ii) 2; (iii) $2a$; (iv) $2a$.
12. (i) $\frac{1}{\sqrt{2}}$; (ii) $\frac{1}{\sqrt{2}}$; (iii) $\frac{2b}{(1+a^2)^{\frac{3}{2}}}$; (iv) $\frac{2b}{(1+a^2)^{\frac{3}{2}}}$.

Exercises. XXI. Page 186.

4. $EIy = \frac{1}{48}W(4x^3 - 12Lx^2 + 9L^2x - L^3)$.

Exercises. XXII. Page 191.

2. $i = i_0e^{-\frac{Rt}{L}}$; $i_1/i_0 = 0{\cdot}368$; $i_2/i_0 = 0{\cdot}135$; $i_3/i_0 = 0{\cdot}050$; $i_4/i_0 = 0{\cdot}018$.
9. (i) 0; (ii) $I/\sqrt{2}$.

Exercises. XXIII. Page 221.

14. 2·137812. **15.** 0·739085. **16.** 1·1656. **17.** 2·588968.

Exercises. XXIV. Page 234.

1. $y=(x^2+C)/(x^2-C)$.
2. $y=Cx/(x-1)$.
3. $y(1+x^2)=C(1-y^2)$.
4. $1+u-v=Ce^{2u}(1-u+v)$.
5. $\log y+x/y=C$.
6. $y^3+3x^2y=C$.
7. $2Cy=x^2-C^2$.
8. $y^2-x=Cy$.
9. $y^2=2Cx+C^2$.
10. $y=\frac{1}{2}\sin x+C/\sin x$.
11. $y=\sqrt{(1+x^2)}[\log\{x+\sqrt{(1+x^2)}\}+C]$.
12. $y=a+C\sqrt{(1-x^2)}$.
13. $y=\sin x+C\cos x$.
14. $pr+\frac{1}{2}r^2=C$.
15. $x=t+Ce^{-t^2}$.
16. $y^2(1-x^2)+x^2-2\log x=C$.
17. $x^2+y^2+a^2=Cx$.
18. $1/y=-a+C\sqrt{(1-x^2)}$.
19. $1/y=\frac{1}{2}(\sin x+\cos x)+Ce^x$.
20. $x(C-t)=\cos t$.
21. $x^2+y^2=(kx+C)^2$.
22. $y(x-C)^2=aC^2$.
23. $y=\frac{a}{2}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)$.
24. $y=Ae^{ax}+Be^{-2ax}$.
25. $x=Ae^{-at}+Be^{-bt}$.
26. $y=Ae^{\frac{x}{2}}+Be^{-2x}+\frac{e^{3x}}{25}$.
27. $x=Ae^{-t/2}+Be^{-t/3}+\frac{1}{2}e^{-t}$.
28. $y=Ae^{2x}+e^{-x}(B\cos x+C\sin x)-\frac{1}{5}e^x$.
29. $y=e^{-x}(A\cos 2x+B\sin 2x)+\frac{1}{17}(\cos 2x+4\sin 2x)$.
30. $y=e^{-3x}(A\cos 3x+B\sin 3x)+2\cos 3x+4\sin 3x$.
31. $y=A\cos 3x+B\sin 3x-\frac{a}{10}\cos 2x+\frac{a}{18}$.
32. $y=A\cos 3x+B\sin 3x-\frac{a}{6}x\cos 3x$.
33. $y=A\cos x+B\sin x+\frac{a}{4}x\sin x+\frac{a}{16}\cos 3x$.
34. $y=e^{-\frac{x}{2}}\left(A\cos\frac{\sqrt{3}}{2}x+B\sin\frac{\sqrt{3}}{2}x\right)-\cos x$.
35. $x=e^{-3t}(A\cos 2t+B\sin 2t)+\frac{a}{75}(3\cos 2t+4\sin 2t)$.
36. $x=A+Be^{-2t}-\frac{1}{2}te^{-2t}+\frac{1}{4}t^2-\frac{1}{4}t$.
37. $y=Ae^{-x}+(B+Cx)e^{2x}+\frac{a}{6}x^2e^{2x}+\frac{b}{9}xe^{-x}$.
38. $y=(A+Bx)e^x+(C+Dx)e^{-x}+x^2+2x+5$.
39. $y=Ae^{\frac{4x}{3}}+Be^{-x}-\frac{1}{4}x^2+\frac{1}{8}x-\frac{13}{32}$.

40. $x = Ae^{at} + Be^{-2at} - \dfrac{bt}{2a^2} - \dfrac{2ac+b}{4a^3}$. **41.** $y = e^x(\sin 3x - 18\cos 3x)/325$.

42. $y = ae^{-mx}(m \sin nx + 2n \cos nx)/(m^3 + 4mn^2)$.

43. $y = \dfrac{ae^x}{15}\left(x^2 - \dfrac{16x}{15} + \dfrac{98}{225}\right)$. **44.** $y = -\frac{1}{2}xe^{-x}\cos x$.

45. (i) $x = (a \sin pt)/(n^2 - p^2)$; (ii) $x = -(at/2n)\cos nt$.

46. (i) $x = \dfrac{ae^{-pt}(p \sin nt + 2n \cos nt)}{p(p^2+4n^2)}$; (ii) $x = \dfrac{ae^{-nt}(\sin nt + 2\cos nt)}{5n^2}$.

47. $x = \dfrac{b}{a}(a+t)e^{-t/a}$. **48.** $x = \dfrac{a}{4}\cos t - \dfrac{a}{4}e^{-2t}(\cos t + 2\sin t)$; $\dfrac{a}{4}\cos t$.

49. $x = (a \sin t - \sin at)/(a^3 - a)$; $\frac{1}{2}(\sin t - t\cos t)$.

50. $x = E\{(a^2+b^2-p^2)\sin pt - 2ap\cos pt\}/\{(a^2+b^2-p^2)^2 + 4a^2p^2\}$.

51. $kx = \log\left(\dfrac{e^{t\sqrt{gk}} + e^{-t\sqrt{gk}}}{2}\right)$; $\sqrt{(g/k)}$.

52. $kx = \log\left(\dfrac{V}{w}\sin\dfrac{gt}{w} + \cos\dfrac{gt}{w}\right)$, $w = \sqrt{(g/k)}$.

53. $l\dot{\theta}^2 = 2g(\cos\theta - \cos\alpha)$.

54. $V = A + B/r^{n-1}$ if $n \neq 1$; $V = A + B\log r$ if $n = 1$.

55. $\dot{x}^2 = 2\mu(x^{-1} - a^{-1})$.

Miscellaneous Examples. PAGE 240.

2. $y = 4x^3 - 9x^2 - 12x + 20$. **3.** $(3ac - b^2)/3a$. **4.** 32 cms. × 18 cms.

5. T.P. $(-2, -13)$, Minimum; P.I. $(-1, 9)$, grad. $= 32$; $(1, 41)$, grad. $= 0$.

8. 0; $h/6$. **9.** $4\sqrt{3}$ (or 6·93) inches per min. **12.** Const. $= -2c$.

13. $x\dot{y} - y\dot{x} = r^2\dot{\theta}$, $x\ddot{x} + y\ddot{y} = r\ddot{r} - (r\dot{\theta})^2$, $x\ddot{y} - y\ddot{x} = d(r^2\dot{\theta})/dt$.

15. 60 nautical miles; after 3 hours.

16. $(ay - x^2)/(y^2 - ax)$. If z is always zero the equation is the equation of a curve, and dy/dx is the gradient at the point (x, y) on the curve.

17. $4(x+2)/9(1-y)$; (i) $(-2, -1)$, $(-2, 3)$; (ii) $(-5, 1)$, $(1, 1)$.

18. $\dfrac{2axy - x(x^2+y^2)}{y(x^2+y^2) - ax^2} = \dfrac{2y(x^2-y^2)}{x(x^2-3y^2)}$; (i) $(0, 0)$, (a, a), $(-a, a)$; (ii) $(0, 0)$, $\left(\dfrac{3\sqrt{3}}{4}a, \dfrac{3}{4}a\right)$, $\left(-\dfrac{3\sqrt{3}}{4}a, \dfrac{3}{4}a\right)$. The origin is a multiple point at which both axes touch the curve.

19. (i) $(2ax - y^2 - 3x^2)/2y(x+a)$; (ii) $(y^2 + 2xy - 2x)/(2y - 2xy - x^2)$.

22. $(3t + 2t^2)/(2+t)$, $2(1+t)^3(3+t)/t(2+t)^3$.

23. $2\log(x-2) - (12x^2 - 39x + 29)/(x-2)^3$; $15\log_e 2 - \frac{665}{64} = 0{\cdot}00658$.

26. $1{\cdot}193k$, but $1{\cdot}198k$ is more correct. **27.** $\frac{4}{3}\pi(R^2 - r^2)^{\frac{3}{2}}$.

28. At K on HB, where $HK = 2{\cdot}656$ inches ; $2{\cdot}238$ (inch units).

29. $4h/7$. **33.** $\frac{2}{3}Ma^2b^2/(a^2+b^2)$. **34.** $2{\cdot}138$ cub. ft.; 27600 ft.-lbs.

35. $\frac{10}{9}\log_e 10 = 2{\cdot}558$. **37.** $k = 27a/8$.

38. $9\pi^2/4$. **39.** $-(1+n\cos t)/a\sin^3 t$.

40. OX ; $\omega a/(b-a)$, $-\omega a/(b+a)$. OY ; $-\omega a^2/(a^2+b^2)$, $\omega a^2/(a^2+b^2)$.

41. $\sec\theta = \sqrt[3]{2}$, $\theta = 37°\ 28'$.

44. $\log(2\cos x + 5\sin x)$; $\lambda = 3$, $\mu = -1$; $3x - \log(2\cos x + 5\sin x)$;
$\frac{7}{13}x - \frac{22}{13}\log(3\cos x - 2\sin x)$;
$\frac{AP+BQ}{P^2+Q^2}x + \frac{AQ-BP}{P^2+Q^2}\log(P\cos x + Q\sin x)$.

45. 2π. **46.** $5\pi a^2$. **47.** 0, $\frac{256}{315}$, $\frac{63\pi}{256}$, $\frac{63\pi}{256}$.

50. $\rho_0(1-\frac{3}{4}k)$. **51.** $\pi/3$, $13/\sqrt{2}$.

53. (i) $\frac{-1-2x^2}{4(1+x^2)^2}$, (ii) $\frac{x^4}{4(1+x^2)^2}$; these differ by the *constant* $\frac{1}{4}$.

54. $\mp 1/x\sqrt{(x^2-1)}$ respectively ; $\frac{1}{2}\sqrt{(\sec x + 1)}$.

55. $\frac{1}{5}\log\tan\left(\frac{x}{2}+\frac{1}{2}\tan^{-1}\frac{3}{4}\right)$; $\frac{1}{\sqrt{(A^2+B^2)}}\log\tan\left(\frac{x}{2}+\frac{1}{2}\tan^{-1}\frac{A}{B}\right)$.

56. $\frac{1}{6}\tan^{-1}\left(\frac{5}{6}\tan\frac{x}{2}+\frac{1}{3}\right)$; $\tan^{-1}\left(\frac{5}{2}\tan x - \frac{3}{2}\right)$.

57. (i) $1{\cdot}0136$; (ii) $0{\cdot}9863$.

58. $-\frac{(1+x^2)^{\frac{3}{2}}}{3x^3}$, $x = \frac{u}{\sqrt{(1-u^2)}}$ or $1+\frac{1}{x^2}=\frac{1}{u^2}=z^2$; $\tan^{-1}\sqrt{(x-1)}$;
$\sin^{-1}(x-1)+\sqrt{(2x-x^2)}$.

59. (i) $2\pi ac$; (ii) $(\pi-2)ac$. **60.** Apply the method of § 44.

61. $56°{\cdot}7$; 19 min. 50 sec. **62.** $-\tan\theta$; constant $= a$.

65. $4a\cos\frac{3}{2}t$. **66.** $-1{\cdot}3$.

67. a ; $x^2+y^2+2ay=0$. **70.** $kab/2\pi$, density $= kxy$; $\left(\frac{8a}{15}, \frac{8b}{15}\right)$.

Miscellaneous Integrals. Page 248.

1. $\sqrt{(x^2-a^2)} - a\tan^{-1}\frac{1}{a}\sqrt{(x^2-a^2)}$. **2.** $2\tan^{-1}(e^x)$.

3. $\pi\{b-\sqrt{(b^2-a^2)}\}/2b$. **4.** $\tan\frac{1}{2}x + \log(1+\cos x)$.

5. $2\sqrt{(x-3)} + \frac{4}{\sqrt{3}}\tan^{-1}\sqrt{\left(\frac{x-3}{3}\right)}$. **6.** π.

7. $\frac{1}{2}(1+x^2)\log(1+x^2) - \frac{1}{2}x^2$. **8.** $1/24a^4$.

9. $\sqrt{2}\cos^{-1}\sqrt{\left(\frac{1-x^2}{1+x^2}\right)} - \sin^{-1}x$.

10. $(b-a)\sin^{-1}\sqrt{\left(\dfrac{x-a}{b-a}\right)}-\sqrt{\{(x-a)(b-x)\}}$. **11.** $3\pi a^2/32$.

12. $(\pi+2)/8$. **13.** $\frac{1}{12}(2\cos 6x-3\cos 4x-6\cos 2x)$.

14. $-\sqrt{(1-x-x^2)}-\dfrac{1}{2}\sin^{-1}\left(\dfrac{2x+1}{\sqrt{5}}\right)$. **15.** $4/25$. **16.** $\log\left(\dfrac{\tan\theta}{1+\tan\theta}\right)$.

17. $\log\{x-2+\sqrt{(x^2-4x+3)}\}$. **18.** $\dfrac{3}{4}a^2\sin^{-1}\sqrt{\left(\dfrac{x}{a}\right)}-\dfrac{3a+2x}{4}\sqrt{(ax-x^2)}$.

19. $\dfrac{a\theta}{a^2+b^2}+\dfrac{b}{a^2+b^2}\log(a\cos\theta+b\sin\theta)$. **20.** $-17\pi/2$.

21. $\frac{1}{5}\tan^{-1}(\frac{1}{5}\tan x)$. **22.** $\dfrac{1}{2\sqrt{2}}\log\dfrac{\sqrt{2}-\sqrt{(1-x^2)}}{\sqrt{2}+\sqrt{(1-x^2)}}$.

23. $\frac{1}{3}\sin^{-1}(x^3)$. **24.** $\pi/2$. **25.** $x\tan^{-1}x-\frac{1}{2}\log(1+x^2)$.

26. $\frac{1}{2}e^x\{(1-x)\cos x+x\sin x\}$. **27.** $\frac{9}{64}(20-9\log 3)$.

28. $1+\sqrt{2}-\dfrac{\pi}{4}$. **29.** $\dfrac{1}{a}\sec^{-1}\left(\dfrac{x-a}{a}\right)$.

30. $\frac{1}{3}\log[\{(x-1)^{\frac{3}{2}}-3x+7\}^2/(x-2)(x-5)^2]$.

31. $\frac{4}{5}\theta+\frac{3}{20}\log\{(2+\tan\frac{1}{2}\theta)/(2-\tan\frac{1}{2}\theta)\}$. **32.** $\pi/8$.

33. $25\left(\dfrac{3\pi}{4}+\dfrac{5}{3}\right)$. **34.** $\frac{1}{2}\log(1-x^2)+x\sin^{-1}x/\sqrt{(1-x^2)}$.

35. $25\pi/6912$. **36.** $\dfrac{1}{2a^2}[x^2+x\sqrt{(x^2-a^2)}-a^2\log\{x+\sqrt{(x^2-a^2)}\}]$.

37. $\frac{1}{2}\tan x\sec x+\frac{1}{2}\log(\tan x+\sec x)$. **38.** $\pi/\sqrt{3}$. **39.** $7/3$.

40. $\{\log(a\cos^2x+b\sin^2x)\}/2(b-a)$.

41. (i) $\log(1+e^\theta)$; (ii) $\theta-\log(1+e^\theta)$; (iii) $e^\theta-\log(1+e^\theta)$;
(iv) $\log(1+e^\theta)-\theta-e^{-\theta}$.

42. $\frac{2}{27}(18-5\sqrt{3})$. **43.** $\dfrac{1}{4a^2}\log\dfrac{a^2+x^2}{a^2-x^2}$. **44.** $\dfrac{2}{\sqrt{15}}\tan^{-1}\left(\dfrac{3+2\tan\frac{1}{2}x}{\sqrt{15}}\right)$.

45. $(3\pi+8)/32a^5$. **46.** $\dfrac{1}{18}\log\left(\dfrac{x^6}{3+5x^6}\right)$. **47.** $\theta/\sin\theta$.

48. $\dfrac{4}{105}(1+x^{\frac{1}{2}})^{\frac{3}{2}}(8-12x^{\frac{1}{2}}+15x)$. **49.** $\dfrac{5}{52}\log\dfrac{x-2}{x+2}+\dfrac{7}{13}\tan^{-1}\dfrac{1}{3}x$.

50. (i) $\dfrac{1}{m\sqrt{(n^2-m^2)}}\tan^{-1}\left\{\dfrac{x\sqrt{(n^2-m^2)}}{m\sqrt{(x^2+n^2)}}\right\}$;

(ii) $\dfrac{1}{2m\sqrt{(m^2-n^2)}}\log\left\{\dfrac{m\sqrt{(x^2+n^2)}+x\sqrt{(m^2-n^2)}}{m\sqrt{(x^2+n^2)}-x\sqrt{(m^2-n^2)}}\right\}$.

INDEX.

The numbers refer to pages.

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